PHASES OF LAGRANGIAN-INVARIANT OBJECTS IN THE DERIVED CATEGORY OF AN ABELIAN VARIETY

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Abstract. We continue the study of Lagrangian-invariant objects (LI-objects for short) in the derived category $D^b(A)$ of coherent sheaves on an abelian variety, initiated in [31]. For every element of the complexified ample cone D_A we construct a natural phase function on the set of LI-objects, which in the case $\dim A = 2$ gives the phases with respect to the corresponding Bridgeland stability (see [9]). The construction is based on the relation between endofunctors of $D^b(A)$ and a certain natural central extension of groups, associated with D_A viewed as a hermitian symmetric space.

Introduction

The notion of stability condition on triangulated categories, introduced by Bridgeland in [8], axiomatizes the notion of stability of branes coming from the study of deformations of superconformal field theories (see [10]). The hope is that the space of stability conditions on a Calabi-Yau threefold are related to the moduli spaces of complex structures on a mirror dual manifold. At present we have examples of Bridgeland stabilities on $D^b(X)$ for any surface X, however, the problem of constructing such examples for a Calabi-Yau threefold is still open (see [2] for a proposal of such a construction).

The goal of this paper is to test the existence of a stability condition on $D^b(A)$ for any abelian variety A by looking at certain special objects in $D^b(A)$. More precisely, for an element $\omega = i\alpha + \beta \in D_A \subset \mathrm{NS}(A) \otimes \mathbb{C}$ in the complexified ample cone (defined by the condition that α is ample) one expects to have a stability condition on $D^b(A)$ with the central charge

$$Z(F) = \int_A \exp(-\omega) \cdot \mathrm{ch}(F),$$

where $F \in D^b(A)$. The starting point of this work is the observation that there are certain objects in $D^b(A)$ that are automatically semistable with respect to any nice stability condition (see Prop. 3.1.4). Namely, these are *Lagrangian-invariant objects* (*LI-objects* for short) defined in [31] (see also Def. 2.1.1). The simplest examples are the structure sheaves of points $\mathcal{O}_x$. To get other examples one can consider images of $\mathcal{O}_x$ under autoequivalences of $D^b(A)$ but in general these do not exhaust all LI-objects (see Remark 4.2.2 and Prop. 4.2.3). Thus, a stability condition should give a *phase* for any LI-object F, i.e., a lifting of $\mathrm{Arg}\, Z(F) \in \mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$. Furthermore, a nonzero morphism $F_1 \to F_2$ can exist only if the phase of F_1 does not exceed the phase of F_2. The main result of this paper is the construction of such a phase function associated with each $\omega \in D_A$. We also verify some properties of this function that one expects from the theory of stability conditions (see Thm. 3.3.2).

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The major role in our construction is played by the action of a certain group on the set $\overline{\mathrm{SH}}^{LI}/\mathbb{N}^*$ of classes of LI-objects modulo certain simple equivalence relation (we allow to apply translations and tensoring with a line bundle in $\mathrm{Pic}^0(A)$ and with a vector space). This group, which we denote $\widetilde{\mathbf{U}(\mathbb{Q})}$ is a central extension by $\mathbb{Z}$ of the group of $\mathbb{Q}$-points of an algebraic group $\mathbf{U} = \mathbf{U}_{X_A}$ defined as automorphisms of the abelian variety $X_A = A \times \hat{A}$, compatible with the skew-symmetric autoduality of X_A. The preimage of the subgroup of $\mathbb{Z}$-points in $\widetilde{\mathbf{U}(\mathbb{Q})}$ is closely related to the group of autoequivalences of $D^b(A)$ (see [21, 25, 23]). The main idea that brings the Siegel domain D_A into picture is that the above central extension has a natural interpretation in terms of the action of $\mathbf{U}(\mathbb{Q})$ on D_A (see Theorem 2.3.2). This allows us to parametrize the set $\overline{\mathrm{SH}}^{LI}/\mathbb{N}^*$ of classes of LI-objects by points of a natural $\mathbb{Z}$-covering of the set of $\mathbb{Q}$-points of a certain homogeneous algebraic variety $\mathbf{LG} = \mathbf{LG}_A$ for the group $\mathbf{U}$ (the points $\mathbf{LG}(\mathbb{Q})$ are in bijection with *Lagrangian* abelian subvarieties in $A \times \hat{A}$), and the phase function appears naturally in this context.

If $\dim A = 2$ then the stability condition corresponding to ω was constructed by Bridgeland in [9], and we check that our phases for LI-objects match the ones coming from this stability condition (see Section 3.4).

In the case when $A = E^n$, where E is an elliptic curve without complex multiplication, we give a mirror-symmetric interpretation of our picture in terms of Fukaya category of the mirror dual abelian variety (following the recipe of [13]). We show that the central charge on LI-objects in $D^b(A)$ defined using $\omega \in D_A$ matches with the integral of the holomorphic volume form over the corresponding Lagrangian tori, and hence, that LI-objects in $D^b(A)$ give rise to graded Lagrangians on the mirror dual side (see Section 3.5).

We also observe that the set $\widetilde{\mathbf{LG}(\mathbb{Q})}$ parametrizing classes of LI-objects also parametrizes certain natural collection of t-structures on $D^b(A)$, generalizing the ones obtained from the standard t-structure by applying autoequivalences (we call them *quasi-standard*). We conjecture that there is also a natural t-structure associated with every point of $\widetilde{\mathbf{LG}(\mathbb{R})}$ whose heart is equivalent to the category of holomorphic bundles on the corresponding noncommutative torus (see [32, 28, 4]).

Another by-product of our study is a refinement of the results of [21, 13] on the action of autoequivalences of $D^b(A)$ on numerical classes of objects. Namely, we construct a natural double covering $\mathrm{Spin} \to \mathbf{U}$ of algebraic groups over $\mathbb{Q}$ and an algebraic representation of Spin on the vector space associated with the numerical Grothendieck group of A, such that the action of elements projecting to $\mathbf{U}(\mathbb{Q})$ is induced by endofunctors of $D^b(A)$ (see Thm. 2.5.3).

The paper is organized as follows. Section 1 contains some auxiliary results not involving derived categories. In particular, we give an interpretation of the index of a nondegenerate line bundle on an abelian variety in terms of the function $\mathrm{Arg}\,\chi$ on the complexified ample cone (see Theorem 1.2.1). We also prove some useful results about the group $\mathbf{U}$ and the variety of Lagrangian subvarieties $\mathbf{LG}$ in $A \times \hat{A}$. In Section 2 we study the central extension $\widetilde{\mathbf{U}(\mathbb{Q})} \to \mathbf{U}(\mathbb{Q})$ coming from a natural 1-cocycle with values

in $\mathcal{O}^*(D_A)$ and its action on LI-objects in $D^b(A)$ and their numerical classes. In Section 3 we parametrize LI-objects (up to certain equivalence) by points of a natural $\mathbb{Z}$-covering $\widetilde{\mathbf{LG}(\mathbb{Q})} \to \mathbf{LG}(\mathbb{Q})$ equipped with an action of $\widetilde{\mathbf{U}(\mathbb{Q})}$, and construct a family of phase functions on $\widetilde{\mathbf{LG}(\mathbb{Q})}$ parametrized by $D_A \times \mathbb{C}$, equivariantly with respect to $\widetilde{\mathbf{U}(\mathbb{Q})}$. We also study the connection with Bridgeland stability conditions on abelian surfaces (see Thm. 3.4.3) and with mirror symmetry (see Sec. 3.5). In Section 4 we construct a family of t-structures on $D^b(A)$ parametrized by $\widetilde{\mathbf{LG}(\mathbb{Q})}$ and study a relation between $\mathbf{LG}(\mathbb{Q})/\mathbf{U}(\mathbb{Q})$ and the Fourier-Mukai partners of A (see Sec. 4.2).

Notations and conventions. We work over a fixed algebraically closed field k. We say that an object F of a k-linear category $\mathcal{C}$ is *endosimple* if $\mathrm{Hom}_{\mathcal{C}}(F,F)=k$. For a scheme X we denote by $D^b(X)$ the bounded derived category of coherent sheaves on X. We say that an object $F \in D^b(X)$ is *cohomologically pure* if there exists a coherent sheaf H such that $F \simeq H[n]$ for $n \in \mathbb{Z}$. We denote by $\mathcal{A}b_{\mathbb{Q}}$ the category of abelian varieties up to an isogeny (i.e., the localization of the category of abelian varieties over k with respect to the class of isogenies). When we want to consider the F-vector space associated with a $\mathbb{Z}$-lattice M, where $F=\mathbb{Q},\mathbb{R}$, or $\mathbb{C}$, as an algebraic variety over F, we denote it by M_F.

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1. Preliminaries

Throughout this paper A denotes an abelian variety over k.

1.1. **Degree, trace and Euler bilinear form.** Recall that for $f \in \mathrm{End}(A)$ one has

$$\deg(f) = \det T_l(f)$$

where $T_l(f)$ is the representation of f on the Tate module $T_l(A)$ for $l \neq \mathrm{char}(k)$ (see [22, Ch. 19, Thm. 4]). Thus, extending deg to a polynomial function

$$\deg : \mathrm{End}(A) \otimes \mathbb{Q} \to \mathbb{Q},$$

homogeneous of degree $2g$, we have

$$\deg(1+tf) = 1 + t \cdot \mathrm{Tr}(f) + O(t^2),$$

where $\mathrm{Tr}(f)$ is given by the trace of the action of f on $T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Furthermore, $f \mapsto \mathrm{Tr}(f \cdot f')$ is a positive definite quadratic form on $\mathrm{End}(A) \otimes \mathbb{Q}$, where $f \mapsto f'$ is the Rosati involution associated with a polarization of A (see [22, Ch. 21, Thm. 1]).

Let us fix a polarization on A and denote by $\mathrm{End}(A)^+ \otimes \mathbb{Q} \subset \mathrm{End}(A) \otimes \mathbb{Q}$ the subspace of elements invariant with respect to the corresponding Rosati involution. Note that the quadratic form $\mathrm{Tr}(f^2)$ on $\mathrm{End}(A)^+ \otimes \mathbb{Q}$ is positive-definite.

Proposition 1.1.1. *An element $f \in \mathrm{End}(A) \otimes \mathbb{C}$ is determined by the polynomial function*

$$\mathrm{End}(A) \otimes \mathbb{C} \to \mathbb{C} : x \mapsto \deg(f-x).$$

Furthermore, if f is invariant with respect to the Rosati involution then it is determined by the restriction of the above function to $\mathrm{End}(A)^+ \otimes \mathbb{C}$.

Proof. We have to check that if $\deg(f_1 - x) = \deg(f_2 - x)$ for all $x \in \mathrm{End}(A)$ then $f_1 = f_2$. Adding to f_1 and f_2 the same element of $\mathrm{End}(A) \otimes \mathbb{C}$ we can assume that f_1 and f_2 are invertible in $\mathrm{End}(A) \otimes \mathbb{C}$. Observe also that $\deg(f_1) = \deg(f_2)$ (this follows by substituting $x = 0$). Thus, we obtain

$$\deg(1 - xf_1^{-1}) = \deg(f_1 - x)\deg(f_1)^{-1} = \deg(f_2 - x)\deg(f_2)^{-1} = \deg(1 - xf_2^{-1}).$$

Considering the linear terms in x we derive

$$\mathrm{Tr}(xf_1^{-1}) = \mathrm{Tr}(xf_2^{-1}).$$

The nondegeneracy of the form $\mathrm{Tr}(fg)$ implies $f_1^{-1} = f_2^{-1}$.

To prove the second statement, we repeat the above argument letting x vary only in $\mathrm{End}(A)^+ \otimes \mathbb{C}$. $\square$

We always use the standard identification

$$\mathrm{NS}(A) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{Q} : L \mapsto \phi_L,$$

where $\mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{Q} \subset \mathrm{Hom}(A, \hat{A}) \otimes \mathbb{Q}$ consists of self-dual homomorphisms. The Euler characteristic defines a polynomial function $\chi : \mathrm{NS}(A) \otimes \mathbb{C} \to \mathbb{C}$ of degree $g = \dim A$, which we also view as a function on $\mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{C}$. One has $\chi^2 = \deg$ (see [22, ch. 16]).

Recall that the Grothendieck group $K_0(A)$ carries the Euler bilinear form

$$\chi([E], [F]) := \sum_i (-1)^i \dim \mathrm{Hom}^i(E, F),$$

where $E, F \in D^b(A)$. We denote by $\mathcal{N}(A)$ the numerical Grothendieck group, i.e., the quotient of $K_0(A)$ by the kernel of this form. $\mathcal{N}(A)$ is a free abelian group of finite rank (see [14, Ex. 19.1.4]). Associating with a line bundle L its class $[L]$ in $\mathcal{N}(A)$ defines a polynomial map between free abelian groups of finite rank

$$\ell : \mathrm{NS}(A) \to \mathcal{N}(A).$$

Therefore, we have the induced polynomial morphism between the corresponding $\mathbb{Q}$-vector spaces

$$\ell : \mathrm{NS}(A)_{\mathbb{Q}} \to \mathcal{N}(A)_{\mathbb{Q}}. \tag{1.1.1}$$

Corollary 1.1.2. *An element $\phi \in \mathrm{NS}(A) \otimes \mathbb{C}$ is determined by the corresponding polynomial function*

$$\mathrm{NS}(A) \to \mathbb{C} : x \mapsto \chi(\ell(\phi), \ell(x)).$$

Proof. Since $\mathrm{NS}(A)$ is Zariski dense in $\mathrm{NS}(A)_{\mathbb{C}}$, it is enough to prove the similar statement with the polynomial function $\chi(\ell(\phi), \ell(\cdot))$ on $\mathrm{NS}(A) \otimes \mathbb{C}$. Note that

$$\chi(\ell(\phi), \ell(x))^2 = \chi(\ell(x - \phi))^2 = \deg(x - \phi)$$

where we view x and ϕ as elements of $\mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{C}$. Let $\phi_0 : A \to \hat{A}$ be a polarization. Then the map $x \mapsto \phi_0^{-1} \circ x$ gives an isomorphism $\mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{Q} \simeq \mathrm{End}(A)^+ \otimes \mathbb{Q}$ (and the corresponding isomorphism of $\mathbb{C}$-vector spaces). Furthermore, this isomorphism rescales deg by the constant $\deg(\phi_0)$. It remains to apply Proposition 1.1.1. $\square$

Remark 1.1.3. When the ground field is $\mathbb{C}$ we can identify $\mathcal{N}(A)\otimes\mathbb{Q}$ with the subspace of algebraic cycles in $H^*(A,\mathbb{Q})$ via the Chern character and $\mathrm{NS}(A)\otimes\mathbb{Q}$ with algebraic cycles in $H^2(A,\mathbb{Q})$. Then ℓ is induced by the exponential map $H^2(A,\mathbb{Q})\to H^*(A,\mathbb{Q})$.

1.2. **Characterization of the index of a line bundle.** Recall that if L is a nondegenerate line bundle on A (i.e., the corresponding map $\phi_L : A\to\hat{A}$ is an isogeny) then its *index* $i(L)$ is defined by the condition $H^i(A,L)=0$ for $i\neq i(L)$. We will use the following recipe for computing $i(L)$: it is the number of positive roots of the polynomial $P(n)=\chi(L\otimes L_0^n)$, where L_0 is an ample line bundle on A (see [22, ch. 16]). The index function $i(\cdot)$ extends uniquely to a $\mathbb{Q}_{>0}$-invariant function on $\mathrm{NS}(A)_{\mathbb{Q}}$.

Let $D_A\subset\mathrm{NS}(A)\otimes\mathbb{C}$ be the complexified ample cone. Note that the function deg and hence χ does not vanish on D_A (see [13, Lem. A.3]). Since D_A is simply connected, there is a unique continuous branch of the argument $\mathrm{Arg}(\chi(x))$ on D_A, such that for $x=iH$, where H is an ample class (an element of the ample cone) we have $\mathrm{Arg}(\chi(iH))=g\pi/2$, where $g=\dim A$. It is easy to see that this branch does not depend on a choice of H. Then for class $x\in\mathrm{NS}(A)\otimes\mathbb{R}$ with $\chi(x)\neq 0$ we can define by continuity the argument $\mathrm{Arg}(\chi(x))$, i.e., we set

$$\mathrm{Arg}(\chi(x))=\lim_{t\to 0+}\mathrm{Arg}(\chi(x+itH)),$$

where H is an ample class. Note that since $\chi(x)$ is real, the number $\mathrm{Arg}(\chi(x))/\pi$ is an integer.

Theorem 1.2.1. *For the continuous branch of* $\mathrm{Arg}(\chi(\cdot))$ *on* D_A*, satisfying* $\mathrm{Arg}(\chi(iH))=g\pi/2$ *(where H is ample), one has*

$$\mathrm{Arg}(\chi(x))=i(x)\pi$$

for every $x\in\mathrm{NS}(A)\otimes\mathbb{Q}$ *with* $\chi(x)\neq 0$.

Proof. First, let us consider the case when x is in the ample cone. For $z\in\mathbb{C}$ we have $\chi(zx)=z^g\cdot\chi(x)$. Thus, varying z on a unit circle from 1 to i we obtain

$$\mathrm{Arg}(\chi(ix))=\mathrm{Arg}(\chi(x))+\frac{g\pi}{2}.$$

Since $\mathrm{Arg}(\chi(ix))=g\pi/2$, we obtain that $\mathrm{Arg}(\chi(x))=0$. Next, assume $x\in\mathrm{NS}(A)\subset\mathrm{NS}(A)\otimes\mathbb{Q}$. Then for any ample class H the polynomial

$$P(t)=\chi(x+tH)$$

has $i(x)$ positive roots, counted with multiplicity (see [22, ch. 16]). Let $0<t_1<\ldots<t_r$ be all the positive roots of $P(t)$. For $t\gg 0$ the class $x+tH$ is ample and so $\mathrm{Arg}\,\chi(x+tH)=0$. Now we are going to decrease t until it reaches zero and observe the change of $\mathrm{Arg}(P(t))=\mathrm{Arg}(\chi(x+tH))$. Note that it can only change when t passes one of the roots t_j. If t_j is a root of $P(t)$ of multiplicity m_j, then for sufficiently small $\epsilon>0$ one has

$$\mathrm{Arg}(P(t_j-\epsilon))=\mathrm{Arg}(P(t_j+\epsilon))+m_j\pi.$$

Adding up the changes we get

$$\mathrm{Arg}(\chi(x))=\mathrm{Arg}(P(0))=i(x)\pi.$$

Since $i(x)$ does not change upon rescaling by a positive rational number, the assertion for any $x\in\mathrm{NS}(A)\otimes\mathbb{Q}$ follows. ∎

Corollary 1.2.2. *For the branch of* $\mathrm{Arg}(\deg(\cdot))$ *on* D_A *normalized by* $\mathrm{Arg}(\deg(iH)) = g\pi$ *one has*

$$\mathrm{Arg}(\deg(x)) = i(x) \cdot 2\pi$$

for any $x \in \mathrm{NS}(A) \otimes \mathbb{Q}$ *such that* $\deg(x) \neq 0$.

We will also need some information on the restriction of $\mathrm{Arg}(\chi(\cdot))$ to lines of the form $iH + \mathbb{R}x \subset D_A$.

Lemma 1.2.3. *(i) For any ample class* $H \in \mathrm{NS}(A) \otimes \mathbb{Q}$ *and any* $x \in \mathrm{NS}^0(A, \mathbb{Q})$ *let us choose any continuous branch of* $t \mapsto \mathrm{Arg}(\chi(iH + tx))$, *where* $t \in \mathbb{R}$. *Then for* $0 \leq t_1 < t_2$ *one has*

$$\mathrm{Arg}(\chi(iH + t_1 x)) - (g - i(x))\frac{\pi}{2} < \mathrm{Arg}(\chi(iH + t_2 x)) < \mathrm{Arg}(\chi(iH + t_1 x)) + i(x)\frac{\pi}{2}. \tag{1.2.1}$$

(ii) For any continuous branch of $\mathrm{Arg}(\deg(\cdot))$ *on* D_A *one has*

$$\mathrm{Arg}(\deg(\omega)) \leq \mathrm{Arg}(\deg(iH)) + g\pi$$

for any $\omega \in D_A$, *where* H *is an ample class.*

Proof. (i) Indeed, the polynomial

$$P(t) = \chi(iH + tx) = i^g \chi(H + \frac{t}{i}x)$$

has all roots purely imaginary, and exactly $i(x)$ of them in the upper half-plane, counted with multiplicity (see [22, ch. 16]). Let us write $P(t) = c \cdot (t - z_1) \cdot \ldots \cdot (t - z_g)$. Since $P(t) \neq 0$ for all $t \in \mathbb{R}$, we can choose for every $j = 1 \ldots, g$ a continuous branch of $t \mapsto \mathrm{Arg}(t - z_j)$ along the real line and use the branch

$$\mathrm{Arg}\, P(t) = \mathrm{Arg}(c) + \mathrm{Arg}(t - z_1) + \ldots + \mathrm{Arg}(t - z_g).$$

Suppose the roots $z_1, \ldots, z_{i(x)}$ are in the upper half-plane while z_j for $j > i(x)$ are in the lower half-plane. Then for each $j > i(x)$ the function $t \mapsto \mathrm{Arg}(t - z_j)$ is strictly decreasing and we have

$$\mathrm{Arg}(t_1 - z_j) - \frac{\pi}{2} < \mathrm{Arg}(t_2 - z_j) < \mathrm{Arg}(t_1 - z_j).$$

On the other hand, for $j \leq i(x)$ we have

$$\mathrm{Arg}(t_1 - z_j) < \mathrm{Arg}(t_2 - z_j) < \mathrm{Arg}(t_1 - z_j) + \frac{\pi}{2}.$$

Summing up over all the roots gives (1.2.1).
(ii) Applying (1.2.1) to $t_1 = 0$ and $t_2 = 1$ we get

$$\mathrm{Arg}(\chi(iH + x)) \leq \mathrm{Arg}(\chi(iH)) + i(x)\frac{\pi}{2} \leq \mathrm{Arg}(\chi(iH)) + g\frac{\pi}{2}.$$

Since $\deg = \chi^2$ on NS, we get the required inequality for points in D_A with rational real and imaginary part. The general case follows by continuity. $\square$

1.3. **The group $\mathbf{U}_{A\times\hat{A}}$.** Recall (see [21], [24], [23], [13]) that with every abelian variety A one can associate an algebraic group $\mathbf{U}=\mathbf{U}_{X_A}$ over $\mathbb{Q}$, where $X_A:=A\times\hat{A}$, as follows. For every $F\subset\mathbb{Q}$ we define the group of F-points $\mathbf{U}(F)$ as a subgroup of invertible elements of the algebra $\mathrm{End}(X_A)\otimes F$ consisting of

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{End}(X_A)\otimes F \quad \text{with } a\in \mathrm{Hom}(A,A)\otimes F, b\in \mathrm{Hom}(\hat{A},A)\otimes F, \text{ etc.},$$

such that

$$g^{-1}=\begin{pmatrix} \hat{d} & -\hat{b}\\ -\hat{c} & \hat{a}\end{pmatrix}\in \mathrm{End}(A\times\hat{A})\otimes F.$$

The arithmetic subgroup

$$\mathbf{U}(\mathbb{Z}):=\mathbf{U}(\mathbb{Q})\cap \mathrm{End}(A\times\hat{A})$$

is closely related to the group of autoequivalences of $D^b(A)$ (see [23]). When we view the matrix element b above as a function on $\mathbf{U}(F)$ we denote it by $b(g)$.

Our point of view is to consider X_A as a "symplectic object" in the category of abelian varieties using the skew-symmetric self-duality $\eta_A : X_A\widetilde{\rightarrow}\hat{X}_A$ associated with the biextension $p_{14}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}^{-1}$ of $X_A\times X_A$ (see [25], [31]). Then elements of $\mathbf{U}(\mathbb{Z})$ are precisely *symplectic automorphisms* of X_A, i.e., automorphisms compatible with η_A. The development of this point of view in [31] was to view elements of $\mathbf{U}(\mathbb{Q})$ as *Lagrangian correspondences* from X_A to itself, which allowed us to define endofunctors of $D^b(A)$ associated with elements of $\mathbf{U}(\mathbb{Q})$ (see [31, Sec. 3] and Sec. 2.1 below).

Note that we have the algebraic subgroup $\mathbf{T}\simeq(\mathrm{End}(A)_{\mathbb{Q}})^*\subset\mathbf{U}$ consisting of diagonal matrices of the form

$$\begin{pmatrix} a^{-1} & 0\\ 0 & \hat{a}\end{pmatrix}.$$

The following facts about the group $\mathbf{U}$ follow easily from Albert's classification of the endomorphism algebras of simple abelian varieties (see [24], [13]).

Lemma 1.3.1. *(i) Let us fix a polarization $\phi: A\to\hat{A}$ and let $\mathbf{Z}\subset\mathbf{T}$ be the algebraic subgroup corresponding to $a\in(\mathrm{End}(A)_{\mathbb{Q}})^*$ such that a lies in the center of $\mathrm{End}(A)_{\mathbb{Q}}$ and $a^{-1}=\phi^{-1}\hat{a}\phi$. Then the group $\mathbf{U}$ is an almost direct product of the semisimple commutant subgroup $S\mathbf{U}$ and of $\mathbf{Z}$.*
(ii) The algebraic group $\mathbf{U}$ is connected, and the Lie group $\mathbf{U}(\mathbb{R})$ is connected (with respect to the classical topology).

We denote by $\mathbf{U}^0\subset\mathbf{U}$ the Zariski open subset given by the inequality $\deg(b(g))\neq 0$. Note that for any $g\in\mathbf{U}^0(\mathbb{R})$ we have $\deg(b(g))>0$ (since the function deg is nonnegative on $\mathrm{Hom}(A,\hat{A})\otimes\mathbb{R}$).

The following condition on a subset of a group was introduced in [34, IV.42] (the term is due to D. Kazhdan).

Definition 1.3.2. Let G be a group. A subset $B\subset G$ is called *big* if for any $g_1,g_2,g_3\in G$ one has

$$B^{-1}\cap Bg_1\cap Bg_2\cap Bg_3\neq\emptyset.$$

This notion is useful because of the following result (part (i) is due to Weil and part (ii) is a more precise version of [26, Lem. 4.2]).

Lemma 1.3.3. *(i) Let $B \subset G$ be a big subset. Then G is isomorphic to the abstract group generated by elements $[b]$ for $b \in B$ subject to the relations $[b_1][b_2] = [b_1 b_2]$ whenever $b_1 b_2 \in B$.*
(ii) Let Z be an abelian group (with the trivial G-action). Let $c, c' : G \times G \to Z$ be a pair of 2-cocycles such that

$$c(b_1, b_2) = c'(b_1, b_2)$$

for any $b_1, b_2 \in B$ with $b_1 b_2 \in B$. Let $p : G_c \to G$ (resp. $p' : G_{c'} \to G$) be the extension of G by Z associated with c (resp., c'), and let $\sigma : G \to G_c$ (resp., $\sigma' : G \to G_{c'}$) be the natural set-theoretic sections. Then there is a unique isomorphism of extensions $i : G_c \to G_{c'}$ such that $i(\sigma(b)) = \sigma'(b)$ (and identical on Z).

Proof. (i) This is [34, IV.42, Lem. 6].
(ii) Note that the subset $p^{-1}(B) \subset G_c$ (resp., $(p')^{-1}(B) \subset G_{c'}$) is big. Thus, we can define a homomorphism $G_c \to G_{c'}$ be requiring that it sends $z\sigma(b)$ to $z\sigma'(b)$ for $b \in B$, provided we check the compatibility with the relations

$$\sigma(b_1)\sigma(b_2) = c(b_1, b_2)\sigma(b_1 b_2),$$

$$\sigma'(b_1)\sigma'(b_2) = c'(b_1, b_2)\sigma'(b_1 b_2),$$

whenever $b_1, b_2, b_1 b_2 \in B$. But this boils down to the equality $c(b_1, b_2) = c'(b_1, b_2)$. ☐

Next, we will show that the subset $\mathbf{U}^0(\mathbb{Q}) \subset \mathbf{U}(\mathbb{Q})$ (resp., $\mathbf{U}^0(\mathbb{R}) \subset \mathbf{U}(\mathbb{R})$) is big. Note that the subset $\mathbf{U}^0(\mathbb{Q}) \cap \mathbf{U}(\mathbb{Z})$ in the arithmetic group $\mathbf{U}(\mathbb{Z})$ is also big (see Remark 1.4.2).

Lemma 1.3.4. *For any field extension $\mathbb{Q} \subset F$ the set $\mathbf{U}(F)$ is Zariski-dense in $\mathbf{U}$. Hence, the subset $\mathbf{U}^0(F) \subset \mathbf{U}(F)$ is big in $\mathbf{U}(F)$.*

Proof. Since $\mathbf{U}$ is connected, density of $\mathbf{U}(F)$ follows from [6, Cor. 18.3]. Thus, for any $g_1, g_2, g_3 \in \mathbf{U}(F)$ the intersection $\mathbf{U}^0 \cap \mathbf{U}^0 g_1 \cap \mathbf{U}^0 g_2 \cap \mathbf{U}^0 g_3$ contains a point of $\mathbf{U}(F)$. ☐

The group $\mathbf{U}$ has two natural parabolic subgroups: $\mathbf{P}^+$ is the intersection of $\mathbf{U}$ with the subgroup of upper-triangular 2×2-matrices in $\mathrm{End}(A \times \hat{A})_{\mathbb{Q}}$, and $\mathbf{P}^-$ is the intersection with the subgroup of lower-triangular matrices. We also denote by $\mathbf{N}^+ \subset \mathbf{P}^+$ (resp. $\mathbf{N}^- \subset \mathbf{P}^-$) the subgroup of strictly upper-triangular (resp. strictly lower-triangular) matrices. Note that both $\mathbf{N}^+$ and $\mathbf{N}^-$ are isomorphic to $\mathrm{NS}(A)_{\mathbb{Q}}$.

Lemma 1.3.5. *Any normal subgroup of $\mathbf{U}(\mathbb{Q})$ containing $\mathbf{P}^-(\mathbb{Q})$ is the entire $\mathbf{U}(\mathbb{Q})$.*

Proof. Since $\mathbf{P}^+(\mathbb{Q})$ is conjugate to $\mathbf{P}^-(\mathbb{Q})$ by an element

$$w_\phi = \begin{pmatrix} 0 & \phi^{-1} \\ -\phi & 0 \end{pmatrix}, \tag{1.3.1}$$

where $\phi : A \to \hat{A}$ is a polarization, it is enough to check that $\mathbf{U}(\mathbb{Q})$ is generated by the subgroups $\mathbf{P}^+(\mathbb{Q})$ and $\mathbf{P}^-(\mathbb{Q})$. We can write any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{U}(\mathbb{Q})$ with invertible a as

$$g = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & \hat{a}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

Finally, any element of $\mathbf{U}^0(\mathbb{Q})$ has form gw_ϕ with g as above. Thus, the statement follows from Lemma 1.3.4. $\square$

1.4. **Action of $\mathbf{U}(\mathbb{Q})$ on Lagrangian subvarieties.** Recall that an abelian subvariety $L \subset X_A = A \times \hat{A}$ is *isotropic* if the composition

$$L \to X_A \xrightarrow{\eta_A} \hat{X}_A \to \hat{L}$$

is zero, where η_A is the standard skew-symmetric self-duality. If in addition $\dim L = \dim A$ then L is called *Lagrangian* (for other equivalent definitions see [31, Sec. 2.2]). In this case η_A induces an isomorphism $X_A/L \simeq \hat{L}$.

To enumerate all Lagrangian abelian subvarieties in X_A it is convenient to work in the semisimple category $\mathcal{A}b_{\mathbb{Q}}$ of abelian varieties up to isogeny. Note that abelian subvarieties of X_A are in natural bijection with subobjects of X_A in the category $\mathcal{A}b_{\mathbb{Q}}$. Thus, we can use a similar notion of a Lagrangian subvariety in $\mathcal{A}b_{\mathbb{Q}}$. Now if $L \subset X_A$ is Lagrangian then we have an isomorphism $X_A \simeq L \oplus \hat{L}$ in $\mathcal{A}b_{\mathbb{Q}}$, which implies that L is isomorphic to A in $\mathcal{A}b_{\mathbb{Q}}$. Thus, we can describe a Lagrangian subvariety (in the category $\mathcal{A}b_{\mathbb{Q}}$) as an image of a morphism $A \to X_A$, i.e., by a pair (x, y), where $x \in \operatorname{End}(A) \otimes \mathbb{Q}$, $y \in \operatorname{Hom}(A, \hat{A}) \otimes \mathbb{Q}$. The isotropy condition is the equation

$$\hat{y}x = \hat{x}y.$$

The existence of a splitting $X_A \to A$ in $\mathcal{A}b_{\mathbb{Q}}$ is equivalent to the condition

$$(\star) \quad (\operatorname{End}(A) \otimes \mathbb{Q})x + (\operatorname{Hom}(\hat{A}, A) \otimes \mathbb{Q})y = \operatorname{End}(A) \otimes \mathbb{Q}.$$

The pairs (x_1, y_1) and (x_2, y_2) define the same subvariety if and only if there exists an automorphism α of A in $\mathcal{A}b_{\mathbb{Q}}$ such that $x_2 = x_1\alpha$, $y_2 = y_1\alpha$. Thus, we obtain an identification of the set of Lagrangian subvarieties in X_A with the set

$$\mathbf{LG}(\mathbb{Q}) := \{(x, y) \mid \hat{y}x = \hat{x}y, (\star)\}/(x, y) \sim (x\alpha, y\alpha), \tag{1.4.1}$$

where $x \in \operatorname{End}(A) \otimes \mathbb{Q}$, $y \in \operatorname{Hom}(A, \hat{A}) \otimes \mathbb{Q}$ and $\alpha \in (\operatorname{End}(A) \otimes \mathbb{Q})^*$. We denote by $(x : y) \in \mathbf{LG}(\mathbb{Q})$ the equivalence class of $(x, y) \in \operatorname{End}(A) \otimes \mathbb{Q} \oplus \operatorname{Hom}(A, \hat{A}) \otimes \mathbb{Q}$.

Fixing a polarization on A we can identify A with $\hat{A}$, so that the dualization gets replaced by the Rosati involution $x \mapsto x'$ on $\mathcal{A} := \operatorname{End}(A) \otimes \mathbb{Q}$. We claim that the set $\mathbf{LG}(\mathbb{Q})$ can be identified with the set of $\mathbb{Q}$-points of a certain homogeneous projective variety $\mathbf{LG}$ for the group $\mathbf{U}$ (a subvariety in the Grassmannian of right rank-1 $\mathcal{A}$-submodules in $\mathcal{A}^2$). Here the action of $\mathbf{U}$ on $\mathbf{LG}$ is induced by the natural action of $\operatorname{End}(X_A)_{\mathbb{Q}}$ on pairs (x, y) (viewed as column vectors). Consider the point $(0 : \phi_0) \in \mathbf{LG}(\mathbb{Q})$, where $\phi_0 : A \to \hat{A}$ is a polarization, (the corresponding Lagrangian is $0 \times \hat{A} \subset X_A$). Note that the stabilizer subgroup of is the subgroup $\mathbf{P}^- \subset \mathbf{U}$ of lower triangular matrices. Thus, we define

$$\mathbf{LG} = \mathbf{LG}_A = \mathbf{U}/\mathbf{P}^-.$$

The fact that the set (1.4.1) is indeed the set of $\mathbb{Q}$-points of $\mathbf{LG}$ follows from the transitivity of the action of $\mathbf{U}(\mathbb{Q})$ on the set of Lagrangian subvarieties that we will prove below (see Prop. 1.4.3).

We start with the following useful result.

Proposition 1.4.1. *For any collection of Lagrangian subvarieties $L_1, \dots, L_r \subset X_A$ there exists an element $g \in \mathbf{U}(\mathbb{Z})$ such that all the Lagrangians $gL_1, \dots, gL_r$ are transversal to $\{0\} \times \hat{A}$.*

Proof. We use an argument similar to the first part of the proof of [31, Thm. 3.2.11]. Consider elements in $\mathbf{U}(\mathbb{Z})$ of the form g^+_{nb} for some polarization $b : \hat{A} \to A$, where $n \in \mathbb{Z}$. Then the condition that $g^+_{nb}L_i$ is transversal to $\{0\} \times \hat{A}$ is equivalent to L_i being transversal to $g^+_{-nb}(\{0\} \times \hat{A}) = \Gamma(-nb)$. By [31, Lem. 2.2.7(ii)], the latter transversality holds for all n except for a finite number. □

Remark 1.4.2. The above Proposition immediately implies that subset $\mathbf{U}^0 \cap \mathbf{U}(\mathbb{Z})$ of the group $\mathbf{U}(\mathbb{Z})$ is big (see Sec. 1.3). Indeed, for any given $g_1, \dots, g_r \in \mathbf{U}(\mathbb{Z})$ consider the Lagrangian subvarieties $L_i = g_i(\{0\} \times \hat{A}) \subset X_A$, $i = 1, \dots, n$. Then we can find $g \in \mathbf{U}(\mathbb{Z})$ such that $gL_i = gg_i(\{0\} \times \hat{A})$ for $i = 1, \dots, n$ are transversal to $\{0\} \times \hat{A}$. Thus, we get $gg_i \in \mathbf{U}^0$ as required. The same proof works for any finite index subgroup $\Gamma \subset \mathbf{U}(\mathbb{Z})$. The fact that $\mathbf{U}^0 \cap \Gamma$ is a big subset of Γ was stated in [26, Lem. 4.3]. However, the proof in *loc. cit.* was not correct: it relied on the absence of compact factors in $S\mathbf{U}(\mathbb{R})$, which is not always the case (see [13, Cor. 5.3.3]).

Lagrangian subvarieties in X_A, transversal to $0 \times \hat{A}$, are all graphs $\Gamma(f)$ of symmetric homomorphisms $f \in \mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{Q}$ (see [31, Ex. 2.2.4]). This corresponds to points of $\mathbf{LG}(\mathbb{Q})$ of the form $(1 : f)$, which are precisely $\mathbb{Q}$-points of a Zariski open subset

$$\mathrm{NS}(A)_{\mathbb{Q}} \simeq \mathbf{N}^- w_\phi \mathbf{P}^- / \mathbf{P}^- \subset \mathbf{LG}, \tag{1.4.2}$$

where w_ϕ is given by (1.3.1), and $\mathbf{N}^- \subset \mathbf{U}$ is the subgroup of strictly lower triangular matrices. In other words, the subset (1.4.2) is just the $\mathbf{N}^-$-orbit of the point $(1 : 0) \in \mathbf{LG}$.

Proposition 1.4.3. *(i) The action of* $\mathbf{U}(\mathbb{Q})$ *on the set of Lagrangian subvarieties in X_A is transitive.*
(ii) The action of $\mathbf{U}(\mathbb{R})$ *on* $\mathbf{LG}(\mathbb{R})$ *is transitive.*

Proof. (i) The subgroup $\mathbf{N}^-(\mathbb{Q}) \simeq \mathrm{NS}(A) \otimes \mathbb{Q}$ acts on the subset $\mathrm{NS}(A)_{\mathbb{Q}} \subset \mathbf{LG}$ by translations, so the corresponding action on the set of $\mathbb{Q}$-points is transitive. By Proposition 1.4.1, any point of $\mathbf{LG}(\mathbb{Q})$ is obtained from a $\mathbb{Q}$-point of this subset by an action of $\mathbf{U}(\mathbb{Z})$, so the required transitivity follows.

(ii) As is well known, it suffices to check triviality of the kernel of the map of Galois cohomology $H^1(\mathbb{R}, \mathbf{P}^-) \to H^1(\mathbb{R}, \mathbf{U})$. Since $\mathbf{P}^-$ is a semi-direct product of $\prod_i \mathrm{GL}_{n_i}(D_i)$ (where D_i are skew-fields) and of $\mathbb{G}_a^n$, in fact, the set $H^1(\mathbb{R}, \mathbf{P}^-)$ is trivial. □

The description (1.4.1) of $\mathbb{Q}$-points of $\mathbf{LG}$ can be extended to a similar description of $\mathbf{LG}(F)$, where $F = \mathbb{R}$ or $\mathbb{C}$, so we can still use homogeneous coordinates $(x : y)$, where $x \in \mathrm{End}(A) \otimes F$, $y \in \mathrm{Hom}(A, \hat{A}) \otimes F$ to describe points of $\mathbf{LG}(F)$.

The complexified ample cone $D_A \subset \mathrm{NS}(A) \otimes \mathbb{C}$ is a hermitian symmetric space (a tube domain) with the group of isometries $\mathbf{U}(\mathbb{R})$ (see [21, Sec. 5],[13, Sec. 8]). Namely, the group $\mathbf{U}(\mathbb{R})$ acts on D_A by

$$g(\omega) = (c + d\omega)(a + b\omega)^{-1}. \tag{1.4.3}$$

This action is well defined since $\deg(a+b\omega)\neq 0$ for $\omega\in D_A$ (see [13, Lem. A3]). Furthermore, it is transitive and the stabilizer of a point $\omega\in D_A$ is a maximal compact subgroup of $\mathbf{U}(\mathbb{R})$ (see [13, Thm. A1]). Also, the natural embedding

$$D_A\hookrightarrow \mathbf{LG}(\mathbb{C}):\omega\mapsto(1:\omega)$$

is $\mathbf{U}(\mathbb{R})$-equivariant.

2. LI-functors and central extensions

2.1. **LI-objects and functors.** Recall that every object $K\subset D^b(A\times A)$ gives rise to a functor of Fourier-Mukai type

$$\Phi_K:D^b(A)\to D^b(A):F\mapsto Rp_{2*}(p_1^*F\otimes^{\mathbb{L}}K),$$

where p_1 and p_2 are projections of $A\times A$ to its factors (we refer to K as the *kernel* of the functor Φ_K). The composition $\Phi_{K_1}\circ\Phi_{K_2}$ corresponds to the convolution of kernels $K_2\circ_A K_1$ (see [19], our notation is as in [30]).

Recall that in [31] we have extended the relation between autoequivalences of $D^b(A)$ and the group $\mathbf{U}(\mathbb{Z})$ (see [25], [23]) to a construction of endofunctors of $D^b(A)$ (given by kernels on $A\times A$) associated with elements of $\mathbf{U}(\mathbb{Q})$, suitably enhanced. Namely, with every element $g\in\mathbf{U}(\mathbb{Q})$ we associate its graph $L(g)\subset X_A\times X_A$, which we view as a Lagrangian subvariety in $X_A\times X_A$ with respect to the symplectic self-duality $(-\eta_A)\times\eta_A$ (see [31, Sec. 3.1]). The corresponding kernel on $A\times A$ is constructed as a generator of the subcategory of $L(g)$-invariants with respect to the action of $X_A\times X_A$ on $D^b(A\times A)$.

More precisely, every Lagrangian subvariety $L\subset X_A$ can be equipped with a line bundle α such that we have an isomorphism of line bundles on $L\times L$

$$\alpha_{l_1+l_2}\otimes\alpha_{l_1}^{-1}\otimes\alpha_{l_2}^{-1}\simeq\mathcal{P}_{p_A(l_1),p_{\hat{A}}(l_2)},\tag{2.1.1}$$

where $p_A:L\to A$ and $p_{\hat{A}}:L\to\hat{A}$ are the projections, and $\mathcal{P}$ is the Poincaré bundle on $A\times\hat{A}$. We refer to (L,α) as *Lagrangian pair*, For every such pair (L,α) there exists a unique up to an isomorphism endosimple coherent sheaf $S_{L,\alpha}$ on A together with an isomorphism

$$(S_{L,\alpha})_{x+p_A(l)}\otimes\mathcal{P}_{x,p_{\hat{A}}(l)}\otimes\alpha_l\simeq(S_{L,\alpha})_x\tag{2.1.2}$$

on $L\times A$ (where $l\in L$, $x\in A$), satisfying certain natural compatibility condition. We view this condition as invariance with respect to the lifting of L to the *Heisenberg groupoid* $\mathbf{H}=\mathbf{H}_A$, acting on $D^b(A)$ (and on $D^b(A\times S)$ for any scheme S). By definition, $\mathbf{H}$ is a Picard groupoid extension of X_A by the stack of line bundles, so its objects over a scheme S are pairs: a point $(x,\xi)\in X_A(S)$ and a line bundle $\mathcal{L}$ on S. The group operation is determined by

$$(x_1,\xi_1)\cdot(x_2,\xi_2)=\mathcal{P}_{x_1,\xi_2}\cdot(x_1+x_2,\xi_1+\xi_2).$$

The action of $(x,\xi)\in X_A(S)\subset\mathbf{H}(S)$ on $D^b(A\times S)$ is given by the functors

$$F\mapsto T_{(x,\xi)}(F)=\mathcal{P}_\xi\otimes t_x^*F,\tag{2.1.3}$$

where $\mathcal{P}_\xi$ is the line bundle on $A\times S$ corresponding to $\xi\in\hat{A}(S)$. A choice of a line bundle α satisfying (2.1.1) gives a lifting of L to a subgroup of $\mathbf{H}$, and the left-hand side of (2.1.2) is the result of the action of $l\in L$ on $S_{L,\alpha}$.

Definition 2.1.1. *LI-objects* are cohomologically pure nonzero objects in $D^b(A)$ that can be equipped with (L,α)-invariance isomorphism (2.1.2) for some (L,α) as above. In fact, they are all of the form $S_{L,\alpha}^{\oplus n}[m]$ for some (L,α), $n\in\mathbb{N}$ and $m\in\mathbb{Z}$ (see [31, Thm. 2.4.5]). Let $\mathrm{SH}^{LI}(A)$ denote the set of isomorphism classes of LI-objects on A. In this work we work mostly with the set $\overline{\mathrm{SH}}^{LI}(A)$ of LI-objects viewed up to the action of $\mathbf{H}(k)$, i.e., up to translations and tensoring with line bundles in $\mathrm{Pic}^0(A)$. We will refer to this equivalence relation as $\mathbf{H}$-equivalence.

We will use the notation $N\cdot F := F^{\oplus N}$ for an LI-object F. This defines an action of the multiplicative monoid $\mathbb{N}^*$ on $\overline{\mathrm{SH}}^{LI}(A)$.

Proposition 2.1.2. *There is a well-defined map*

$$\mathbf{LG}(\mathbb{Q})\to\overline{\mathrm{SH}}^{LI}(A): L\mapsto S(L)$$

sending a Lagrangian subvariety $L\subset X_A$ to the class of the LI-sheaf $S_{L,\alpha}$, where (L,α) is a Lagrangian pair extending L. The map

$$\mathbf{LG}(\mathbb{Q})\times\mathbb{N}^*\times\mathbb{Z}\to\overline{\mathrm{SH}}^{LI}(A):(L,N,n)\mapsto N\cdot S(L)[n]$$

is a bijection of $\mathbb{N}^*\times\mathbb{Z}$-sets.

Proof. The fact that $S(L)$ depends only on L follows from [31, Lem. 2.4.2]. The second statement follows from [31, Thm. 2.4.5] about the structure of the category of (L,α)-invariants in $D^b(A)$ and [31, Cor. 2.4.11] stating that L can be recoved from $S_{L,\alpha}$. $\square$

Recall that for an element $\phi\in\mathrm{NS}(A)\otimes\mathbb{Q}\simeq\mathrm{Hom}(A,\hat{A})^+\otimes\mathbb{Q}$ the graph $\Gamma(\phi)$ is a Lagrangian subvariety of X_A. Furthermore, these graphs are precisely all Lagrangians $L\subset X_A$ such that the projection $L\to A$ is an isogeny. The sheaf $S_{\Gamma(\phi),\alpha}$ associated with a Lagrangian pair $(\Gamma(\phi),\alpha)$, is a simple semihomogeneous vector bundle with $c_1/\mathrm{rk}=\phi$ (see [20]). For $\phi\in\mathrm{NS}(A)\otimes\mathbb{Q}$ we denote the $\mathbf{H}$-equivalence class of this bundle by

$$V_\phi = S(\Gamma(\phi)). \tag{2.1.4}$$

The above construction of LI-sheaves can be applied to Lagrangian subvarieties $L\subset X_A\times X_B$ for a pair of abelian varieties A and B, where we use the symplectic self-duality $(-\eta_A)\times\eta_B$ of $X_A\times X_B$. We refer to the corresponding Lagrangian pairs (L,α) as *Lagrangian correspondences from X_A to X_B*. The obtained LI-sheaves $S_{L,\alpha}$ on $A\times B$ can be used as kernels of *LI-functors*

$$\Phi_{L,\alpha} := \Phi_{S_{L,\alpha}} : D^b(A)\to D^b(B).$$

The key property of these functors is that we have canonical isomorphisms

$$\Phi_{L,\alpha}\circ T_{p_1(l)}\simeq \alpha_l\otimes T_{p_2(l)}\circ\Phi_{L,\alpha} \tag{2.1.5}$$

for $l\in L$, where $p_1,p_2: L\to X_A$ are two projections. Note that every exact equivalence $D^b(A)\to D^b(B)$ is given by such an LI-functor with L being the graph of a symplectic isomorphism $X_A\simeq X_B$ (see [23]).

Let $p_{AB}: L\to A\times B$, $p_{A\hat{A}}: L\to A\times\hat{A}$ and $p_{B\hat{B}}: L\to B\times\hat{B}$ be the projections. The line bundle α can always be chosen in such a way that its restriction to the connected

component of zero in $\ker(p_{AB})$ is trivial. In this case $S_{L,\alpha}$ is a direct summand in

$$p_{AB*}\left(\alpha^{-1}\otimes p_{A\hat{A}}^{*}\mathcal{P}^{-1}\otimes p_{B\hat{B}}^{*}\mathcal{P}\right) \tag{2.1.6}$$

(see [31, Lem. 3.2.5]). In the case when p_{AB} is an isogeny the finite group scheme $\ker(p_{AB})$ has a canonical central extension H_L by $\mathbb{G}_m$ with the underlying line bundle $\alpha|_{\ker(p_{AB})}$. Furthermore, H_L is a Heisenberg group scheme and (2.1.6) has a natural H_L-action, so that

$$S_{L,\alpha}=p_{AB*}\left(\alpha^{-1}\otimes p_{A\hat{A}}^{*}\mathcal{P}^{-1}\otimes p_{B\hat{B}}^{*}\mathcal{P}\right)^{I}, \tag{2.1.7}$$

for a maximal isotropic subgroup $I\subset\ker(p_{AB})$ lifted to H_L. It follows from the theory of weight one representations of Heisenberg groups that taking I-invariants reduces rank by the factor of $|\ker(p_{AB})|^{1/2}$, so we get

$$\operatorname{rk} S_{L,\alpha}=\deg(p_{AB}:L\to A\times B)^{1/2}. \tag{2.1.8}$$

In particular, for $B=0$ we get

$$\operatorname{rk} V_{\phi}=\det(p_A:\Gamma(\phi)\to A)^{1/2}. \tag{2.1.9}$$

Example 2.1.3. The functor of tensoring with a line bundle L on $D^b(A)$ commutes with the action of $\hat{A}$ and satisfies

$$L\otimes(t_x^*F)\simeq\mathcal{P}_{-\phi_L(x)}\otimes t_x^*(L\otimes F).$$

In fact, it is the LI-functor corresponding to $g_{-\phi_L}=\begin{pmatrix}\mathrm{id} & 0\\ -\phi_L & \mathrm{id}\end{pmatrix}$. More generally, for $\phi\in\mathrm{NS}(A)\otimes\mathbb{Q}$ the LI-functor corresponding to the element $g_{-\phi}\in\mathbf{N}^-(\mathbb{Q})$ is the functor of tensoring with the semihomogeneous vector bundle V_ϕ (up to $\mathbf{H}$-equivalence).

The above construction gives a map

$$\mathbf{U}(\mathbb{Q})\to\overline{\mathrm{SH}}^{LI}(A\times A):g\to S(g)=S(L(g)). \tag{2.1.10}$$

We denote by $\Phi_g\in\mathrm{Fun}(D^b(A),D^b(A))/\mathbf{H}$ the functor associated with the kernel $S(g)$, defined up to composing with a functor of the form $T_{(x,\xi)}$, $(x,\xi)\in X_A$ (on either side). For each $(x,\xi)\in X_A$ we have (noncanonical) isomorphisms

$$\Phi_g\circ T_{N(x,\xi)}\simeq T_{Ng(x,\xi)}\circ\Phi_g,$$

where N is such that $Ng\in\mathrm{End}(X_A)$.

Note that we have a well defined homomorphism induced by Φ_g

$$\rho(g):\mathcal{N}(A)\to\mathcal{N}(A).$$

Definition 2.1.4. Let F be a cohomologically pure object of $D^b(A)$ and let G be an endosimple LI-object. We write

$$F\equiv N\cdot G$$

if there exists $n\in\mathbb{Z}$ such that $F[n]$ and $G[n]$ are sheaves and $F[n]$ has a filtration of length N such that each consecutive quotient is $\mathbf{H}$-equivalent to $S(g_1g_2)$. In the case of sheaves on $A\times A$ we will use the same notation for the relation between the corresponding endofunctors of $D^b(A)$.

One of the main results of [31] is the following calculation of the convolution of kernels (see [31, Thm. 3.3.4]):

$$S(g_2)\circ_A S(g_1)\equiv N(g_1,g_2)\cdot S(g_1g_2)[\lambda(g_1,g_2)], \tag{2.1.11}$$

for some 2-cocycles $N(g_1,g_2)$ and $\lambda(g_1,g_2)$ of $\mathbf{U}(\mathbb{Q})$ with values in $\mathbb{N}^*$ and $\mathbb{Z}$, respectively. [1] Furthermore, we have

$$N(g_1,g_2)=\frac{q(L(g_1))^{1/2}q(L(g_2))^{1/2}}{q(L(g_1g_2))^{1/2}}, \tag{2.1.12}$$

where

$$q(g)=q(L(g))=\deg(p_1:L(g)\to X_A). \tag{2.1.13}$$

Also, for $g_1,g_2\in\mathbf{U}^0(\mathbb{Q})$ such that $g_1g_2\in\mathbf{U}^0(\mathbb{Q})$ one has

$$\lambda(g_1,g_2)=-i(b(g_1)^{-1}b(g_1g_2)b(g_2)^{-1}). \tag{2.1.14}$$

Note that in order for the right-hand side to be well-defined the argument of $i(\cdot)$ should be symmetric. This indeed follows from the equality

$$b_1^{-1}(a_1b_2+b_1d_2)b_2^{-1}=b_1^{-1}a_1+d_2b_2^{-1},$$

where we use the usual notation for the entries of g_1 and g_2.

Definition 2.1.5. We denote by $\widetilde{\mathbf{U}(\mathbb{Q})}$ the central extension of $\mathbf{U}(\mathbb{Q})$ by $\mathbb{Z}$ associated with the 2-cocycle $\lambda(\cdot,\cdot)$. Explicitly $\widetilde{\mathbf{U}(\mathbb{Q})}=\mathbf{U}(\mathbb{Q})\times\mathbb{Z}$ with the product

$$(g_1,n_1)\cdot(g_2,n_2)=(g_1g_2,n_1+n_2+\lambda(g_1,g_2)).$$

Note that since the subset $\mathbf{U}^0(\mathbb{Q})\subset\mathbf{U}(\mathbb{Q})$ is big (see Lemma 1.3.4), by Lemma 1.3.3(ii), the formula (2.1.14) determines the extension $\widetilde{\mathbf{U}(\mathbb{Q})}$ uniquely up to a unique isomorphism.

Let us denote by $\overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^*$ the set of equivalence classes with respect to the equivalence relation generated by $F\sim N\cdot F$ for some $N\in\mathbb{N}^*$. By (2.1.11), the map $g\mapsto S(g)\bmod\mathbb{N}^*$ defines a homomorphism of monoids

$$\widetilde{\mathbf{U}(\mathbb{Q})}\to\overline{\mathrm{SH}}^{LI}(A\times A)^{op}/\mathbb{N}^*, \tag{2.1.15}$$

and hence a homomorphism of monoids

$$\widetilde{\mathbf{U}(\mathbb{Q})}\to\mathrm{Fun}(D^b(A),D^b(A))/(\mathbf{H}\times\mathbb{N}^*):g\mapsto\Phi_g, \tag{2.1.16}$$

where on the right we consider functors up to $\mathbf{H}$-equivalence and up to replacing Φ with $N\cdot\Phi=\Phi^{\oplus N}$.

On the level of numerical Grothendieck groups we can eliminate taking quotient by $\mathbb{N}^*$. Namely, let us set for $g\in\mathbf{U}(\mathbb{Q})$

$$\hat{\rho}(g)=\frac{\rho(g)}{q(g)^{1/2}}:\mathcal{N}(A)\otimes\mathbb{R}\to\mathcal{N}(A)\otimes\mathbb{R}. \tag{2.1.17}$$

[1]In [31, Thm. 3.3.4] we made the assumption $\mathrm{char}(k)=0$ which implies a stronger statement: the left-hand side of (2.1.11) is a direct sum of objects $\mathbf{H}$-equivalent to the right-hand side. It is easy to see that the same argument in the positive characteristic case gives a filtration instead of a direct sum.

Then from (2.1.11) and (2.1.12) we derive that

$$\hat{\rho}(g_1)\hat{\rho}(g_2) = (-1)^{\lambda(g_1,g_2)}\hat{\rho}(g_1g_2),$$

where $g_1, g_2 \in \mathbf{U}(\mathbb{Q})$. Thus, $\hat{\rho}$ defines a homomorphism from $\widetilde{\mathbf{U}(\mathbb{Q})}$ to $\mathrm{GL}(\mathcal{N}(A)\otimes\mathbb{R})$, trivial on the central subgroup $2\mathbb{Z} \subset \mathbb{Z} \subset \widetilde{\mathbf{U}(\mathbb{Q})}$. Note that the quotient $\widetilde{\mathbf{U}(\mathbb{Q})}/2\mathbb{Z}$ is a double cover of $\mathbf{U}(\mathbb{Q})$. Below we will introduce an algebraic structure on this double cover and will show that $\hat{\rho}$ is induced by an algebraic homomorphism defined over $\mathbb{R}$ (see Sections 2.3 and 2.5).

2.2. **Splittings over subgroups.** We are going to define a splitting of the central extension $\widetilde{\mathbf{U}(\mathbb{Q})} \to \mathbf{U}(\mathbb{Q})$ over the parabolic subgroup $\mathbf{P}^+(\mathbb{Q}) \subset \mathbf{U}(\mathbb{Q})$ of lower-triangular matrices (resp., over the subgroup $\mathbf{P}^-(\mathbb{Q})$ of upper-triangular matrices). Note that $\mathbf{P}^+(\mathbb{Q})$ is a semidirect product of the subgroups of strictly upper triangular matrices $\mathbf{N}^+(\mathbb{Q}) \simeq \mathrm{NS}(A)\otimes\mathbb{Q}$ and of diagonal matrices $\mathbf{T}(\mathbb{Q}) \simeq (\mathrm{End}(A)\otimes\mathbb{Q})^*$.

Proposition 2.2.1. *(i) There exist unique liftings of the subgroups* $\mathbf{N}^+(\mathbb{Q})$ *and* $\mathbf{N}^-(\mathbb{Q})$ *to* $\widetilde{\mathbf{U}(\mathbb{Q})}$. *The lifting of the element* $g_\phi^+ = \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$, *where* $\phi \in \mathrm{NS}^0(A,\mathbb{Q})$ *is given by* $(g_\phi^+, i(\phi)) \in \widetilde{\mathbf{U}(\mathbb{Q})}$. *The lifting of the element* $g_\phi^- = \begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}$, *where* $\phi \in \mathrm{NS}^0(A,\mathbb{Q})$ *is given by* $(g_\phi^-, 0) \in \widetilde{\mathbf{U}(\mathbb{Q})}$. *The corresponding functor* $\Phi_{g_\phi^-}$ *(defined up to* $\mathbf{H}$*-equivalence) is given by tensoring with the semihomogeneous bundle* $V_{-\phi}$ *(see* (2.1.4)*).*
(ii) For $t = t_a = \begin{pmatrix} a^{-1} & 0 \\ 0 & \hat{a} \end{pmatrix} \in \mathbf{T}(\mathbb{Q})$ *we have (up to* $\mathbf{H}$*-equivalence)*

$$S(t) = \mathcal{O}_B$$

for some abelian subvariety $B \subset A \times A$ *such that the two projections* $p, q : B \to A$ *are isogenies. Hence, the functor* Φ_t *is of the form* q_*p^* *(up to* $\mathbf{H}$*-equivalence).*
(iii) For any $t \in \mathbf{T}(\mathbb{Q})$ *and* $g \in \mathbf{U}(\mathbb{Q})$ *one has* $\lambda(t, g) = 0$.

Proof. (i) Uniqueness of liftings follows from the fact that there are no non-trivial homomorphisms from a $\mathbb{Q}$-vector space to $\mathbb{Z}$. Thus, to check the formula for the lifting of g_ϕ^+ we have to check that

$$S(g_{\phi_2}^+) \circ S(g_{\phi_1}^+) = S(g_{\phi_1+\phi_2}^+)[i(\phi_1+\phi_2) - i(\phi_1) - i(\phi_2),$$

for $\phi_1, \phi_2 \in \mathrm{NS}^0(A,\mathbb{Q})$ such that $\phi_1 + \phi_2 \in \mathrm{NS}^0(A,\mathbb{Q})$. But

$$\lambda(g_{\phi_1}^+, g_{\phi_2}^+) = -i(\phi_1^{-1}(\phi_1+\phi_2)\phi_2^{-1}) = -i(\phi_1^{-1}+\phi_2^{-1}),$$

so we are reduced to showing that

$$i(\phi_1^{-1}+\phi_2^{-1}) = i(\phi_1) + i(\phi_2) - i(\phi_1+\phi_2).$$

Since

$$i(\phi_1^{-1}+\phi_2^{-1}) = i(\phi_1(\phi_1^{-1}+\phi_2^{-1})\phi_1) = i(\phi_1 + \phi_1\phi_2^{-1}\phi_1),$$

this follows from [27, Prop. 15.8] (taking into account that $i(-x) = g - i(x)$).

Since the composition of functors $\otimes V_{\phi_1}$ and $\otimes V_{\phi_2}$ is again tensoring with a bundle that has a filtration with consecutive quotients $\mathbf{H}$-equivalent to $V_{\phi_1+\phi_2}$, the assertion about the lifting of g_ϕ^- follows (cf. Ex. 2.1.3).

(ii) Assume first that $a \in \mathrm{End}(A)$. Then $L(t_a) \simeq A \times \hat{A}$ and its embedding into $X_A \times X_A$ is given by

$$(x, \xi) \mapsto (ax, \xi, x, \hat{a}\xi).$$

This implies that $(L(t_a), \mathcal{O})$ is a Lagrangian correspondence from X_A to itself, so (2.1.6) in this case gives that

$$S_{L(t_a),\mathcal{O}} \simeq (a, \mathrm{id}_A)_* \mathcal{O}_A$$

and the corresponding functor Φ_{t_a} is the pull-back functor a^*. Similarly, if $a^{-1} \in \mathrm{End}(A)$ then

$$S_{L(t_a),\mathcal{O}} \simeq (\mathrm{id}_A, a^{-1})_* \mathcal{O}_A$$

and the corresponding functor Φ_{t_a} is the push-forward functor $(a^{-1})_*$. The general case is obtained by combining these two.

(iii) We have to check that the convolution $S(g) \circ_A S(t)$ is a sheaf. Indeed, using the form of $S(t)$ from (ii) we obtain

$$S(g) \circ_A S(t) \simeq (\mathrm{id}_A \times q)_* (\mathrm{id}_A \times p)^* S(g),$$

where $p, q : B \to A$ are isogenies. $\square$

Corollary 2.2.2. *There is a unique splitting of the central extension $\widetilde{\mathbf{U}(\mathbb{Q})} \to \mathbf{U}(\mathbb{Q})$ over $\mathbf{P}^+(\mathbb{Q}) \subset \mathbf{U}(\mathbb{Q})$ (resp., over $\mathbf{P}^-(\mathbb{Q})$), which maps $t \in \mathbf{T}(\mathbb{Q})$ to $(t, 0) \in \widetilde{\mathbf{U}(\mathbb{Q})}$.*

2.3. **Identifying central extensions.** Recall that $D_A \subset \mathrm{NS}(A) \otimes \mathbb{C} \simeq \mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{C}$ denotes the complexified ample cone of A.

Consider the function $\Delta : \mathbf{U}(\mathbb{R}) \to \mathcal{O}^*(D_A)$ given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Delta(g)(\omega) = \deg(a + b\omega),$$

where $\omega \in D_A$.

Lemma 2.3.1. *For $g_1, g_2 \in \mathbf{U}(\mathbb{R})$ one has*

$$\Delta(g_1 g_2)(\omega) = \Delta(g_1)(g_2(\omega)) \cdot \Delta(g_2)(\omega), \tag{2.3.1}$$

i.e., Δ is a 1-cocycle.

Proof. This follows from the identity

$$a + b\omega = (a_1 + b_1 g_2(\omega))(a_2 + b_2\omega),$$

where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ for $i = 1, 2$ and $g_1 g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\square$

Since D_A is contractible, we have an exact sequence of $\mathbf{U}(\mathbb{R})$-modules

$$0 \to \mathbb{Z} \to \mathcal{O}(D_A) \xrightarrow{\exp(2\pi i \cdot ?)} \mathcal{O}^*(D_A) \to 0.$$

Applying the boundary homomorphism $H^1(\mathbf{U}(\mathbb{R})) \to H^2(\mathbf{U}(\mathbb{Z}))$ to the 1-cocycle $\Delta(g)^{-1}$ we obtain a central extension U^Δ of $\mathbf{U}(\mathbb{R})$ by $\mathbb{Z}$. Explicitly,

$$U^\Delta = \{(g,f) \in \mathbf{U}(\mathbb{R}) \times \mathcal{O}(D_A) \mid \Delta(g) = \exp(-2\pi i f)\}.$$

The multiplication rule on U^Δ uses the cocycle condition on Δ: we set

$$(g_1, f_1) \cdot (g_2, f_2) = (g_1 g_2, f_1(g_2(\cdot)) + f_2).$$

Theorem 2.3.2. *There is a homomorphism $\iota : \widetilde{\mathbf{U}(\mathbb{Q})} \to U^\Delta$, lifting the natural embedding $\mathbf{U}(\mathbb{Q}) \to \mathbf{U}(\mathbb{R})$ and sending $n \in \mathbb{Z} \subset \widetilde{\mathbf{U}(\mathbb{Q})}$ to $(1,n) \in U^\Delta$. This homomorphism is uniquely characterized by the condition that for $g \in \mathbf{U}^0(\mathbb{Q})$ one has*

$$\iota(g, 0) = (g, f),$$

where

$$\lim_{n\to\infty} \operatorname{Re} f(inH) = -\frac{g}{2}$$

for any ample class H.

Proof. First, we are going to define a section $\sigma : \mathbf{U}^0(\mathbb{R}) \to U^\Delta$ of the projection $U^\Delta \to \mathbf{U}(\mathbb{R})$ over the open subset $\mathbf{U}^0(\mathbb{R}) \subset \mathbf{U}(\mathbb{R})$ consisting of g with $\deg(b(g)) \neq 0$. Note that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{U}^0(\mathbb{R})$ one has

$$\Delta(g)(\omega) = \deg(a + b\omega) = \deg(b) \cdot \deg(b^{-1}a + \omega).$$

Since $\deg(b) > 0$, to define $\sigma(g) = (g, f_g^\sigma)$ amounts to choosing a branch of the argument for $\deg(b^{-1}a + \omega)^{-1}$. Let us choose the branch of the argument of $\deg(b^{-1}a + \omega)$ in such a way that

$$\lim_{n\to+\infty} \operatorname{Arg}(\deg(b^{-1}a + inH)) = \pi \cdot g,$$

where H is an ample class and set $\operatorname{Arg}(\Delta(g)(\omega)^{-1}) = -\operatorname{Arg}(\deg(b^{-1}a+\omega))$. Then we set $\iota(g,0) = \sigma(g)$ for $g \in \mathbf{U}^0(\mathbb{Q})$. Since $\mathbf{U}^0(\mathbb{Q})$ is big in $\mathbf{U}(\mathbb{Q})$, by Lemma 1.3.3, it remains to show that for $g_1, g_2 \in \mathbf{U}^0(\mathbb{Q})$ such that $g_1 g_2 \in \mathbf{U}^0(\mathbb{Q})$ one has

$$\sigma(g_1)\sigma(g_2) = \sigma(g_1 g_2) \cdot (1, \lambda(g_1, g_2)).$$

In other words, we have to check that

$$f_{g_1}^\sigma(g_2(\omega)) + f_{g_2}^\sigma(\omega) = f_{g_1 g_2}^\sigma(\omega) + \lambda(g_1, g_2),$$

or equivalently, that with the above choice of $\operatorname{Arg}(\Delta(g))$ one has

$$\operatorname{Arg}(\Delta(g_1)(g_2(\omega)) + \operatorname{Arg}(\Delta(g_2)(\omega)) = \operatorname{Arg}(\Delta(g_1 g_2)(\omega)) - 2\pi \cdot \lambda(g_1, g_2). \tag{2.3.2}$$

It is enough to check the equality of the limits of both sides for $\omega = inH$ as n goes to infinity (where H is an ample class). Let $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ for $i = 1, 2$. Note that

$$\lim_{n\to\infty} g_2(inH) = d_2 b_2^{-1}.$$

Thus, (2.3.2) reduces to the equality

$$\operatorname{Arg}(\Delta(g_1)(d_2 b_2^{-1})) = -2\pi\lambda(g_1, g_2) = i(b_1^{-1} b(g_1 g_2) b_2^{-1}).$$

But

$$\operatorname{Arg}(\Delta(g_1)(d_2b_2^{-1})) = \operatorname{Arg}(\deg(b_1^{-1}a_1 + d_2b_2^{-1}) = \operatorname{Arg}(\deg(b_1^{-1}b(g_1g_2)b_2^{-1}) = 2\pi \cdot i(b_1^{-1}b(g_1g_2)b_2^{-1})$$

by Corollary 1.2.2. □

The central extension $U^\Delta \to \mathbf{U}(\mathbb{R})$ has a natural continuous splitting over the subgroup $\mathbf{P}^-(\mathbb{R}) \subset \mathbf{U}(\mathbb{R})$. Indeed, for $g \in \mathbf{P}^-(\mathbb{R})$ we have $\Delta(g) = \deg(a) > 0$, so we can lift g to

$$\sigma_{\mathbf{P}^-}(g) = (g, -\frac{1}{2\pi i}\log(\deg(a))),$$

where we choose $\log(\deg(a))$ to be in $\mathbb{R}$. The following result will be useful for us later.

Lemma 2.3.3. *The restriction of the above lifting homomorphism $\mathbf{P}^-(\mathbb{R}) \to U^\Delta$ to $\mathbf{P}^-(\mathbb{Q})$ corresponds via ι to the lifting homomorphism $\mathbf{P}^-(\mathbb{Q}) \to \widetilde{\mathbf{U}(\mathbb{Q})}$ considered in Corollary 2.2.2.*

Proof. By Proposition 2.2.1(i), it is enough to check the compatibility of liftings on $\mathbf{T}(\mathbb{Q})$. In view of Proposition 2.2.1(iii) this follows from the equality

$$\sigma_{\mathbf{P}^-}(t)\sigma(g) = \sigma(tg)$$

for any $g \in \mathbf{U}^0(\mathbb{Q})$, where $\sigma : \mathbf{U}^0(\mathbb{R}) \to U^\Delta$ is the section used in the proof of Theorem 2.3.2. □

Similarly, the extension $U^\Delta \to \mathbf{U}(\mathbb{R})$ has a natural continuous splitting over $\mathbf{P}^+(\mathbb{R})$, which is the same as before over $\mathbf{T}(\mathbb{R})$, and over $\mathbf{N}^+(\mathbb{R})$ is described as follows.

Lemma 2.3.4. *There is a unique splitting of $U^\Delta \to \mathbf{U}(\mathbb{R})$ over $\mathbf{N}^+(\mathbb{R}) \simeq \mathrm{NS}(\hat{A}, \mathbb{R})$ which is given by the branch of*

$$\operatorname{Arg}\Delta^{-1}|_{\mathbf{N}^+(\mathbb{R})} = \operatorname{Arg}\deg(1 + \psi\omega)$$

that tends to 0 as $\omega \to 0$, where $\psi \in \mathrm{NS}(\hat{A}, \mathbb{R}) \simeq \mathrm{Hom}(\hat{A}, A)^+_{\mathbb{R}}$.

Proof. It is straightforward to check that this choice of argument gives a lifting. The uniqueness follows from the fact that there are no nontrivial homomorphisms from a real vector space to $\mathbb{Z}$. □

Let us consider the induced double cover $U^\Delta/2\mathbb{Z} \to \mathbf{U}(\mathbb{R})$. We are going to introduce an algebraic structure on this group.

Lemma 2.3.5. *Consider a field extension $\mathbb{Q} \subset F$, where either $F = \mathbb{R}$ or F is algebraically closed. Then for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{U}(F)$, the polynomial $\Delta(g)(\phi) = \deg(a + b\phi)$ on $\mathrm{NS}(A)(F)$ is a complete square (and is nonzero).*

Proof. For $g \in \mathbf{U}^0$ this follows from the equality

$$\deg(a + b\phi) = \deg(b)\deg(b^{-1}a + \phi) = \deg(b)\chi(b^{-1}a + \phi)^2$$

and the fact that $\deg(b) \geq 0$ in the case $F = \mathbb{R}$. Viewing the equation (2.3.1) as an identity of rational functions on $\mathrm{NS}(A)$, we see that if $\Delta(g_1)$ and $\Delta(g_2)$ are complete

squares then $\Delta(g_1g_2)$ is a complete square as a rational function on $\mathrm{NS}(A)$, and hence, as a polynomial. $\square$

Definition 2.3.6. Let $\mathrm{Pol}_{\le g}(\mathrm{NS}(A))$ denote the space of polynomials of degree $\le g$ on $\mathrm{NS}(A)$. We define a double covering $\mathrm{Spin} = \mathrm{Spin}_{X_A} \to \mathbf{U}$ of algebraic groups over $\mathbb{Q}$ by setting

$$\mathrm{Spin} = \{(g, f) \in \mathbf{U} \times \mathrm{Pol}_{\le g}(\mathrm{NS}(A)) \mid \Delta(g) = f^2\}$$

with the group law

$$(g_1, f_1) \cdot (g_2, f_2) = (g_1g_2, f_1(g_2(\cdot)) \cdot f_2).$$

Here the rational function $f_1(g_2(\cdot)) \cdot f_2$ is actually a polynomial since its square is $\Delta(g_1g_2)$.

Note that by Lemma 2.3.5, the map $\pi : \mathrm{Spin}(\mathbb{R}) \to \mathbf{U}(\mathbb{R})$ is a double covering. We have a natural isomorphism of groups

$$U^\Delta/2\mathbb{Z} \to \mathrm{Spin}(\mathbb{R}) : (g, f) \mapsto (g, \exp(-\pi i f)). \tag{2.3.3}$$

We have two natural subgroups in $\mathrm{Spin}(\mathbb{R})$:

$$\mathbf{U}(\mathbb{Q})^{\mathrm{spin}} = \pi^{-1}(\mathbf{U}(\mathbb{Q})), \quad \mathbf{U}(\mathbb{Z})^{\mathrm{spin}} = \pi^{-1}(\mathbf{U}(\mathbb{Z})). \tag{2.3.4}$$

Lemma 2.3.7. *Consider the homomorphism*

$$\bar{\iota} : \widetilde{\mathbf{U}(\mathbb{Q})}/2\mathbb{Z} \xrightarrow{\sim} \mathbf{U}(\mathbb{Q})^{\mathrm{spin}} \subset \mathrm{Spin}(\mathbb{R})$$

induced by $\iota : \widetilde{\mathbf{U}(\mathbb{Q})} \to U^\Delta$ *(see Theorem 2.3.2) and the isomorphism* (2.3.3)*. Then for* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{U}^0(\mathbb{Q})$ *we have*

$$\bar{\iota}(g, 0) = (g, \sqrt{\deg(b)} \cdot \chi(b^{-1}a + \phi)),$$

with $\sqrt{\deg(b)} > 0$.

Proof. By Theorem 2.3.2, $\bar{\iota}(g, 0) = (g, f)$, where $f(\phi)$ is the square root of $\Delta(g)(\phi) = \deg(b) \cdot \deg(b^{-1}a + \phi)$ with the property

$$\lim_{n\to+\infty} \operatorname{Arg} f(inH) = \frac{\pi \cdot g}{2} \bmod 2\pi\mathbb{Z}.$$

Since $\operatorname{Arg} \chi(b^{-1}a + inH)$ has the same limit as $n \to +\infty$, the assertion follows. $\square$

Remarks 2.3.8. 1. If for a field extension $\mathbb{Q} \subset F$ there is a multiplicative norm Nm on $\mathrm{End}(A) \otimes F$ such that $\mathrm{Nm}^2 = \deg$ then the map $g \mapsto (g, \mathrm{Nm}(a + b\omega))$ defines a splitting of the extension $\mathrm{Spin} \to \mathbf{U}$ over F. For example, if $A = E^n$, where E is an elliptic curve without complex multiplication, then $\mathrm{End}(A) = \mathrm{Mat}_n(\mathbb{Z})$ and $\deg([M]_A) = \det(M)^2$ for a matrix $M \in \mathrm{Mat}_n(\mathbb{Z})$. Hence, in this case the norm $\det(\cdot)$ gives a splitting of the spin-covering.
2. The group $\mathbf{U}(\mathbb{Z})^{\mathrm{spin}}$ is exactly the group $USpin(A \times \hat{A})$ defined by Mukai in [21] (the same group is denoted by $\mathrm{Spin}(A)$ in [13]).

Using the isomorphism $\widetilde{\mathbf{U}(\mathbb{Q})}/2\mathbb{Z} \simeq \mathbf{U}(\mathbb{Q})^{\text{spin}}$ we can define a homomorphism

$$\hat{\rho} : \mathbf{U}(\mathbb{Q})^{\text{spin}} \to \mathrm{GL}(\mathcal{N}(A) \otimes \mathbb{R}) \tag{2.3.5}$$

such that $\hat{\rho}(\bar{\iota}(g, 0))$ is the operator $\hat{\rho}(g)$ (see (2.1.17)).

2.4. **The action on LI-objects.** Recall that with a Lagrangian correspondence from X_A to itself extending a symplectic isomorphism $g : X_A \to X_A$ in $\mathcal{A}b_{\mathbb{Q}}$ we associate an endofunctor Φ_g of $D^b(A)$, defined up to $\mathbf{H}$-equivalence (see Sec. 2.1). We are going to use these endofunctors to define an action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on some extension of $\overline{\mathrm{SH}}^{LI}(A)$ (see Corollary 2.4.2).

Theorem 2.4.1. *(i) For an element $g \in \mathbf{U}(\mathbb{Q})$ and a Lagrangian subvariety $L \subset X_A$ we have*

$$\Phi_g(S(L)) \equiv N(g, L) \cdot S(gL)[\lambda(g, L)] \tag{2.4.1}$$

for some $\lambda(g, L) \in \mathbb{Z}$ and $N(g, L) \in \mathbb{N}^*$, where we use Def. 2.1.4.

(ii) If $L = \Gamma(\phi)$ for an isogeny $\phi \in \mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{Q}$ and if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $\deg(b) \neq 0$, $\deg(a + b\phi) \neq 0$ and $\deg(c + d\phi) \neq 0$, then we have

$$N(g, L) = \deg(a + b\phi)^{1/2} \cdot q(g)^{1/2} \cdot \frac{\mathrm{rk}\, V_{g(\phi)}}{\mathrm{rk}\, V_\phi}, \tag{2.4.2}$$

where $q(g)$ is given by (2.1.13), $\mathrm{rk}\, V_\phi$ *is given by* (2.1.9), *and*

$$\lambda(g, \Gamma(\phi)) = -i(b^{-1}a + \phi). \tag{2.4.3}$$

Proof. (i) Let us extend L and $L(g)$ to Lagrangian pairs (L, α) and $(L(g), \beta)$. By [31, Thm. 3.2.11], applied to the Lagrangian correspondence $(L(g), \beta)$ and to (L, α) viewed as a Lagrangian correspondence from 0 to X_A, we obtain

$$\Phi_{L(g),\beta}(S_{L,\alpha}) = S_{L(g)\circ L, \beta\circ\alpha}[i]$$

for some $i \in \mathbb{Z}$. As in [31, Thm. 3.2.14] one can check that i does not depend on α and β. Next, we have to relate the composed Lagrangian correspondence $S_{L(g)\circ L,\beta\circ\alpha}$ with $S(gL)$. Here we use the definition of the composition of Lagrangian correspondences from [31, Sec. 3]. Note that the result is a *generalized Lagrangian correspondence* in the sense of [31, Def. 3.1.1] We are going to apply [31, Prop. 2.4.7(ii)] to the generalized Lagrangian $Z := L(g) \circ L \xrightarrow{j} X_A$. Note that $Z \subset L(g) \subset X_A \times X_A$ is the preimage of L under the first projection $p_1 : L(g) \to X_A$, and the homomorphism $j : Z \to X_A$ is induced by the second projection $p_2 : L(g) \to X_A$. By [31, Prop. 2.4.7(ii)], we have

$$S_{Z,\beta\circ\alpha} \equiv n^{1/2} \cdot |\pi_0(Z)|^{1/2} \cdot S(j(Z_0))$$

in $\overline{\mathrm{SH}}^{LI}(A \times A)$, where $n = |\pi_0(j(Z))|$ (here Z_0 is the connected component of 0 in Z). By definition, we have $j(Z_0) = gL$. Thus, we deduce (2.4.1) with

$$N(g, L) = |\pi_0(Z)|^{1/2} \cdot n^{1/2}.$$

Also, by [31, (2.4.12)], we have $n = \deg(Z_0 \to j(Z_0))$.

(ii) Now assume that $L = \Gamma(\phi)$ and that $g(\phi)$ is defined and is an isogeny. Note that for sufficiently divisible N we have an isogeny

$$A \to Z_0 : x \mapsto (Nx, N\phi x, N(a+b\phi)x, N(c+d\phi)x) \in L(g) \subset X_A \times X_A. \tag{2.4.4}$$

In particular, both projections from Z_0 to A are isogenies. Let us consider the commutative diagram of isogenies

$$\begin{tikzcd} Z_0 \arrow[r] \arrow[d] & Z \arrow[d, "p_{A,2}"'] \\ j(Z_0) \arrow[r, "p_A"] & A \end{tikzcd}$$

where $p_{A,2}$ is the composition $Z \to L(g) \xrightarrow{p_2} X_A \xrightarrow{p_A} A$. Considering the degrees we obtain

$$\deg(p_{A,2} : Z \to A) = |\pi_0(Z)| \cdot \deg(p_{A,2}|_{Z_0}) = |\pi_0(Z)| \cdot \deg(j(Z_0) \to A) \cdot n.$$

Recall that $j(Z_0) = gL$, so we get

$$N(g, L) = \frac{\deg(p_{A,2} : Z \to A)^{1/2}}{\deg(gL \to A)^{1/2}}.$$

Now let us consider the projection $p_{A,1} : Z \to L(g) \xrightarrow{p_1} X_A \xrightarrow{p_A} A$. Using the isogeny (2.4.4) we see that

$$Np_{A,2}|_{Z_0} = N(a+b\phi)p_{A,1}|_{Z_0}.$$

Hence,

$$\frac{\deg(p_{A,2} : Z \to A)}{\deg(p_{A,1} : Z \to A)} = \frac{\deg(p_{A,2}|_{Z_0})}{\deg(p_{A,1}|_{Z_0})} = \deg(a+b\phi). \tag{2.4.5}$$

Note that $p_{A,1}$ factors through the projection $Z \to L$ and we have a cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & L(g) \arrow[d, "p_1"'] \\ L \arrow[r] & X_A \end{tikzcd}$$

which shows that $\deg(Z \to L) = \deg(p_1 : L(g) \to X_A) = q(L(g))$. Thus,

$$\deg(p_{A,1} : Z \to A) = q(L(g)) \cdot \deg(L \to A)$$

and (2.4.5) can be rewritten as

$$\deg(p_{A,2} : Z \to A) = \deg(a+b\phi) \cdot q(L(g)) \cdot \deg(L \to A).$$

Therefore,

$$N(g,L) = \deg(a+b\phi)^{1/2} \cdot q(L(g))^{1/2} \frac{\deg(L \to A)^{1/2}}{\deg(gL \to A)^{1/2}}.$$

Recalling that $\operatorname{rk} S(L) = \deg(L \to A)^{1/2}$ we obtain (2.4.2).

Finally, to compute $\lambda(g, \Gamma(\phi))$ we apply [31, Prop. 3.2.9]. Namely, we have to consider the fibered product $\Gamma(\phi) \times_A L(g)$ where we use the first projection $L(g) \to A$. Note that we have an isogeny

$$A \times \hat{A} \to (\Gamma(\phi) \times_A L(g))_0 : (x, \xi) \mapsto ((Nx, N\phi x), (Nx, N\xi, N(ax+b\xi), N(cx+d\xi))), \tag{2.4.6}$$

where N is sufficiently divisible. Next, we set

$$F = \ker(\Gamma(\phi) \times_A L(g) \xrightarrow{\gamma} A),$$

where γ is induced by the projection to $L(g)$ followed by $L(g) \xrightarrow{p_2} X_A \to A$. Note that the composition of γ with the isogeny (2.4.6) is given by $(x, \xi) \mapsto N(ax + b\xi)$. Hence, we have an isogeny

$$A \to F_0 : x \mapsto ((Nx, N\phi x), (Nx, -Nb^{-1}ax, 0, N(cx - db^{-1}ax))). \tag{2.4.7}$$

By [31, Prop. 3.2.9], we have

$$\lambda(g, \Gamma(\phi)) = -i(g_0 \circ f_0^{-1}),$$

where $f_0 : F_0 \to A$ is the natural projection and for $(l, m) \in F_0 \subset \Gamma(\phi) \times L(g)$,

$$g_0(l, m) = p_{\hat{A}}(l) - p_{\hat{A},1}(m),$$

where $p_{\hat{A}} : \Gamma(\phi) \to \hat{A}$ is the natural projection and $p_{\hat{A},1}$ is the composition $L(g) \xrightarrow{p_1} X_A \to \hat{A}$. Thus, the compositions of f_0 and g_0 with the isogeny (2.4.7) are $x \mapsto Nx$ and $x \mapsto N(\phi + b^{-1}ax)$, respectively. Hence,

$$g_0 \circ f_0^{-1} = \phi + b^{-1}a$$

as required. □

Let us set

$$\overline{\mathrm{SH}}^{LI}(A)_{\mathbb{R}} = \overline{\mathrm{SH}}^{LI}(A) \times \mathbb{R}_{>0}/\mathbb{N}^*,$$

where $n \in \mathbb{N}^*$ acts by $(F, r) \mapsto (nF, n^{-1}r)$. Then the bijection of Proposition 2.1.2 extends to a bijection of $\mathbb{R}_{>0} \times \mathbb{Z}$-sets

$$\mathbf{LG}(\mathbb{Q}) \times \mathbb{R}_{>0} \times \mathbb{Z} \xrightarrow{\sim} \overline{\mathrm{SH}}^{LI}(A)_{\mathbb{R}}.$$

Corollary 2.4.2. *There is an action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $\overline{\mathrm{SH}}^{LI}(A)_{\mathbb{R}}$, commuting with $\mathbb{R}_{>0}$-action, such that (g, n) acts by*

$$F \mapsto q(L(g))^{-1/2} \cdot \Phi_g(F)[n].$$

For $g_1, g_2 \in \mathbf{U}(\mathbb{Q})$ and $L \in \mathbf{LG}(\mathbb{Q})$ we have

$$\lambda(g_1, g_2(L)) + \lambda(g_2, L) = \lambda(g_1, g_2) + \lambda(g_1 g_2, L).$$

Also, the maps Φ_g induce an action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $\overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^* \simeq \mathbf{LG}(\mathbb{Q}) \times \mathbb{Z}$.

Note that the natural maps

$$\overline{\mathrm{SH}}^{LI}(A) \to \mathcal{N}(A) \otimes \mathbb{Q}/\mathbb{Q}_{>0} \tag{2.4.8}$$

$$\overline{\mathrm{SH}}^{LI}(A)_{\mathbb{R}} \to \mathcal{N}(A) \otimes \mathbb{R} \tag{2.4.9}$$

associating with an LI-sheaf F its class $[F]$ in $\mathcal{N}(A)$ are $\widetilde{\mathbf{U}(\mathbb{Q})}$-equivariant, where the action on $\mathcal{N}(A) \otimes \mathbb{R}$ is given by $\hat{\rho}$ (see (2.1.17)).

2.5. **Action of** Spin **on** $\mathcal{N}(A)_{\mathbb{R}}$**.** We are going to define an algebraic action of Spin on $\mathcal{N}(A)_{\mathbb{R}}$ inducing the homomorphism $\hat{\rho}$ on $\mathbf{U}(\mathbb{Q})^{\mathrm{spin}} \subset \mathrm{Spin}(\mathbb{R})$. The idea is to use the algebraicity of the corresponding projective representation and of the action of an open subset on a fixed nonzero vector. We will need the following simple result.

Lemma 2.5.1. *Let V be a vector space over a field F, X a scheme (resp., a set), $\overline{f} : X \to \mathbb{P}(V)$, $\overline{g} : X \to \mathbb{P}(V)$ and $\overline{h} : X \to \mathbb{P}(V)$ be regular morphisms (resp., maps to the set of F-points) such that the lines $\overline{f}(x)$,$\overline{g}(x)$ and $\overline{h}(x)$ are all distinct and $\overline{h}(x) \subset \mathrm{span}(\overline{f}(x), \overline{g}(x))$ for each $x \in X$. Suppose we have a lifting of $\overline{f}$ to a regular morphism (resp., map to the set of F-points) $f : X \to V - \{0\}$. Then there exist unique liftings of $\overline{g}$ and $\overline{h}$ to regular morphisms (resp., maps to the set of F-points) $g, h : X \to V - \{0\}$ such that $h = f + g$.*

Proof. Consider a subvariety

$$Y \subset (V - \{0\}) \times (V - \{0\}) \times (V - \{0\})$$

consisting of $(v_1, v_2, v_1 + v_2)$ such that v_1 and v_2 are linearly independent, and a subvariety

$$\overline{Y} \subset (V - \{0\}) \times \mathbb{P}(V) \times \mathbb{P}(V)$$

consisting of (v, L, L') such that $v \notin L$, $v \notin L'$, $L \neq L'$ and $v \in L + L'$. Then the natural projection $p : Y \to \overline{Y}$ is an isomorphism. We have a regular morphism (resp., map to the set of F-points) $(f, \overline{g}, \overline{h}) : X \to \overline{Y}$. Now the components of the corresponding map $X \to Y$ give the required liftings. □

Lemma 2.5.2. *For a symmetric isogeny $\phi \in \mathrm{NS}^0(A, \mathbb{Q})$ we have*

$$\frac{[V_\phi]}{\mathrm{rk}\, V_\phi} = \ell(\phi) \in \mathcal{N}(A) \otimes \mathbb{Q},$$

where V_ϕ is the semihomogeneous vector bundle (2.1.4) *and $\ell : \mathrm{NS}(A) \otimes \mathbb{Q} \to \mathcal{N}(A) \otimes \mathbb{Q}$ is the polynomial map* (1.1.1)*.*

Proof. Since $\mathrm{rk}\, \ell(\phi) = 1$, it suffices to check that the required identity up to proportionality. Recall that if $(L = \Gamma(\phi), \alpha)$ is a Lagrangian pair then the line in $\mathcal{N}(A) \otimes \mathbb{Q}$ corresponding to V_ϕ is spanned by the class of $p_{A*}(\mathcal{L})$, where $p_A : L \to A$ is the projection and $\mathcal{L} = \alpha^{-1} \otimes \mathcal{P}|_L$ (see (2.1.7)). Also, by the definition of a Lagrangian pair,

$$\Lambda(\alpha)_{l_1, l_2} \simeq \mathcal{P}_{p_A(l_1), p_{\hat{A}}(l_2)},$$

so $\phi_{\mathcal{L}} : L \to \hat{L}$ is given

$$\phi_{\mathcal{L}} = \widehat{p_A} \circ p_{\hat{A}} = \widehat{p_{\hat{A}}} \circ p_A,$$

where $p_{\hat{A}} : L \to \hat{A}$ is the projection. Note that for sufficiently divisible N we have an isogeny

$$i : A \to L : x \mapsto (Nx, N\phi x)$$

and the classes of $p_{A*}(\mathcal{L})$ and $[N]_*(i^*\mathcal{L})$ in $\mathcal{N}(A) \otimes \mathbb{Q}$ are proportional. We have

$$\phi_{i^*(\mathcal{L})} = \hat{i} \circ \phi_{\alpha^{-1} \otimes \mathcal{P}|_L} \circ i = \widehat{p_A \circ i} \circ p_{\hat{A} \circ i} = N^2 \phi.$$

Thus, the class $[i^*(\mathcal{L})] \in \mathcal{N}(A) \otimes \mathbb{Q}$ is proportional to $\ell(N^2\phi) = [N]^*\ell(\phi)$. Hence, the class of $[N]_*(i^*\mathcal{L})$ is proportional to

$$[N]_*[N]^*\ell(\phi) = N^{2g}\ell(\phi)$$

as required. $\square$

Theorem 2.5.3. *The homomorphism* $\hat{\rho} : \mathbf{U}(\mathbb{Q})^{\mathrm{spin}} \to \mathrm{GL}(\mathcal{N}(A) \otimes \mathbb{R})$ *(see* (2.1.17)*) extends to an algebraic homomorphism*

$$\hat{\rho} : \mathrm{Spin} \to \mathrm{GL}(\mathcal{N}(A)_{\mathbb{R}})$$

defined over $\mathbb{R}$*. For* $(g, f) \in \mathrm{Spin}(\mathbb{C})$ *and* $\phi \in \mathrm{NS}^0(A, \mathbb{C})$*, such that* $g(\phi)$ *is defined and belongs to* $\mathrm{NS}^0(A, \mathbb{C})$*, we have*

$$\hat{\rho}(g, f)(\ell(\phi)) = f(\phi) \cdot \ell(g(\phi)). \tag{2.5.1}$$

Proof. First, we observe that Theorem 2.4.1 implies (2.5.1) in the case when $(g, f) \in \mathbf{U}(\mathbb{Q})^{\mathrm{spin}} \subset \mathrm{Spin}(\mathbb{R})$ with $g \in \mathbf{U}^0(\mathbb{Q})$ and $\phi \in \mathrm{NS}^0(A, \mathbb{Q})$ is such that $g(\phi)$ is defined and belongs to $\mathrm{NS}^0(A, \mathbb{Q})$. Indeed, from (2.4.1), (2.4.3), (2.4.2) and Lemma 2.5.2 we obtain in this case

$$\hat{\rho}(g)(\ell(\phi)) = (-1)^{i(b^{-1}a+\phi)} |\deg(a + b\phi)|^{1/2} \cdot \ell(g(\phi)) = \deg(b)^{1/2} \cdot \chi(b^{-1}a + \phi) \cdot \ell(g(\phi)).$$

Thus, our claim holds for

$$\bar{\iota}(g, 0) = (g, \deg(b)^{1/2} \cdot \chi(b^{-1}a + \phi)).$$

It remains to note that both sides of (2.5.1) change sign when (g, f) gets multiplied by $-1 \in \{\pm 1\} \subset \mathrm{Spin}$.

By [26, Thm. 5.1], there is an algebraic homomorphism $\mathbf{U} \to \mathrm{PGL}(\mathcal{N}(A)_{\mathbb{Q}})$ sending $g \in \mathbf{U}(\mathbb{Q})$ to $\rho(g) \bmod \mathbb{Q}^*$. Let us denote the corresponding action of $\mathbf{U}$ on $\mathbb{P}(\mathcal{N}(A)_{\mathbb{R}})$ by

$$\bar{\kappa} : \mathbf{U} \times \mathbb{P}(\mathcal{N}(A)_{\mathbb{R}}) \to \mathbb{P}(\mathcal{N}(A)_{\mathbb{R}}).$$

We also have a map

$$\kappa^{\mathbb{Q}} : \mathbf{U}(\mathbb{Q})^{\mathrm{spin}} \times \mathcal{N}(A) \otimes \mathbb{R} \to \mathcal{N}(A) \otimes \mathbb{R} : (\widetilde{g}, \mathbf{v}) \mapsto \hat{\rho}(\widetilde{g})(\mathbf{v})$$

inducing the restriction of $\bar{\kappa}(\mathbb{R})$ to $\mathbf{U}(\mathbb{Q})^{\mathrm{spin}} \times \mathbb{P}(\mathcal{N}(A) \otimes \mathbb{R})$. We are going to extend $\kappa^{\mathbb{Q}}$ to an algebraic morphism using the density of $\mathbf{U}(\mathbb{Q})$ in $\mathbf{U}$ (see Lemma 1.3.4).

Note that for a fixed isogeny ϕ the right-hand side of (2.5.1) extends to a regular morphism (defined over $\mathbb{Q}$)

$$\kappa_{\ell(\phi)} : \pi^{-1}(V) \to \mathcal{N}(A)_{\mathbb{R}},$$

where $V \subset \mathbf{U}^0 \subset \mathbf{U}$ is an open subset of $g \in \mathbf{U}^0$ such that $g(\phi)$ is defined and is an isogeny. Furthermore, as we have seen in the beginning of the proof, the corresponding map on

$\pi^{-1}(V(\mathbb{Q}))$ coincides with the restriction of $\kappa^{\mathbb{Q}}$ to $\pi^{-1}(V(\mathbb{Q}))\times\{\ell(\phi)\}$. In particular, the map

$$\overline{\kappa}_{\ell(\phi)}:\pi^{-1}(V)\to\mathbb{P}(\mathcal{N}(A)_{\mathbb{R}})$$

obtained from $\kappa_{\ell(\phi)}$ is the composition of the projection to V with the restriction of $\overline{\kappa}$ to $\mathbf{U}\times\{\langle\ell(\phi)\rangle\}$ (since we know this on the dense subset $V(\mathbb{Q})$).

Now if $\mathbf{v}\in\mathcal{N}(A)\otimes\mathbb{Q}$ is any vector, linearly independent with $\ell(\phi)$, then by Lemma 2.5.1, we obtain unique liftings

$$\kappa_{\mathbf{v}},\kappa_{\ell(\phi)+\mathbf{v}}:\pi^{-1}(V)\to\mathcal{N}(A)_{\mathbb{R}} \tag{2.5.2}$$

of the restrictions of $\overline{\kappa}$ to $\pi^{-1}(V)\times\{\langle\mathbf{v}\rangle\}$ and $\pi^{-1}(V)\times\{\langle\ell(\phi)+\mathbf{v}\rangle\}$, such that

$$\kappa_{\ell(\phi)+\mathbf{v}}(\widetilde{g})=\kappa_{\ell(\phi)}(\widetilde{g})+\kappa_{\mathbf{v}}(\widetilde{g}).$$

Furthermore, the set-theoretic part of Lemma 2.5.1 implies that the maps (2.5.2) induce the corresponding restrictions of $\kappa^{\mathbb{Q}}$ on $\pi^{-1}(V(\mathbb{Q}))$.

Thus, if we consider a basis of $\mathcal{N}(A)\otimes\mathbb{Q}$ of the form $(\ell(\phi),\mathbf{v}_1,\dots,\mathbf{v}_n)$ then combining the maps $\kappa_{\mathbf{v}_i}$ constructed above we get a regular morphism

$$\hat{\rho}_V:\pi^{-1}(V)\to\mathrm{GL}(\mathcal{N}(A)_{\mathbb{R}})$$

inducing $\hat{\rho}$ on $\pi^{-1}(V(\mathbb{Q}))$. We can cover Spin with open subsets of the form $\pi^{-1}(V)\widetilde{g}$ with $\widetilde{g}\in\mathbf{U}(\mathbb{Q})^{\mathrm{spin}}$ and define a regular morphism $\pi^{-1}(V)\widetilde{g}\to\mathrm{GL}(\mathcal{N}(A)_{\mathbb{R}})$ by sending $\widetilde{h}\widetilde{g}$ to $\hat{\rho}_V(\widetilde{h})\hat{\rho}(\widetilde{g})$. Using the density of $\mathbf{U}(\mathbb{Q})^{\mathrm{spin}}$ in Spin, one easily checks that these maps glue into the required algebraic homomorphism $\pi^{-1}(V)\to\mathrm{GL}(\mathcal{N}(A)_{\mathbb{R}})$. □

Consider the action of $\mathrm{Spin}(\mathbb{R})$ on the trivial $\mathbb{C}^*$-bundle $D_A\times\mathbb{C}^*$ over the domain D_A given by

$$(g,f)\cdot(\omega,z)=(g(\omega),f(\omega)\cdot z),$$

where $(g,f)\in\mathrm{Spin}(\mathbb{R})$, $\omega\in D_A$, $z\in\mathbb{C}^*$. The map $\ell:D\to\mathcal{N}(A)\otimes\mathbb{C}$ (see (1.1.1)) extends to a $\mathbb{C}^*$-equivariant map

$$\ell:D_A\times\mathbb{C}^*\to\mathcal{N}(A)\otimes\mathbb{C}:(\omega,z)\mapsto z\cdot\ell(\omega). \tag{2.5.3}$$

From the identity (2.5.1) we immediately get the following result.

Corollary 2.5.4. *The map* (2.5.3) *is* $\mathrm{Spin}(\mathbb{R})$*-equivariant.*

Proposition 2.5.5. *For any $x,y\in\mathcal{N}(A)\otimes\mathbb{C}$ and any $\widetilde{g}\in\mathrm{Spin}(\mathbb{C})$ one has*

$$\chi(\hat{\rho}(\widetilde{g})(x),\hat{\rho}(\widetilde{g})(y))=\chi(x,y). \tag{2.5.4}$$

Proof. Note that the left-hand side of (2.5.4) depends only on the image of $\widetilde{g}$ in $\mathbf{U}(\mathbb{C})$. Let us first consider the case when this image is an element $g\in\mathbf{U}(\mathbb{Q})$. Consider the functor $\Phi=\Phi_{L(g),\alpha}:D^b(A)\to D^b(A)$ associated with some Lagrangian correspondence $(L(g),\alpha)$ extending g, so that Φ represents the $\mathbf{H}$-equivalence class of Φ_g. Let Ψ be the right adjoint functor to Φ. By [31, Prop. 3.2.7], Ψ differs by a shift from the LI-functor associated with some Lagrangian correspondence extending $L(g^{-1})$. Applying (2.1.11) and (2.1.12) for $g_1=g$ and $g_2=g^{-1}$ we obtain

$$\Psi\circ\Phi\equiv N\cdot\mathrm{Id},$$

where $N = q(g)^{1/2}q(g^{-1})^{1/2}$. Since for $F, G \in D^b(A)$ we have an isomorphism

$$\mathrm{Hom}^*(\Phi(F), \Phi(G)) = \mathrm{Hom}^*(F, \Psi\Phi(G)),$$

we deduce the equality

$$\chi(\hat{\rho}(g)([F]), \hat{\rho}(g)([G])) = \frac{q(g^{-1})^{1/2}}{q(g)^{1/2}} \cdot \chi([F], [G]). \tag{2.5.5}$$

Since $\mathbf{U}(\mathbb{Q})$ is dense in $\mathbf{U}$ (see Lemma 1.3.4), there exists an algebraic character $\varpi : \mathbf{U} \to \mathbb{G}_m$ such that

$$\chi(\hat{\rho}(g)(x), \hat{\rho}(g)(y)) = \varpi(g) \cdot \chi(x, y)$$

for any $g \in \mathbf{U}(\mathbb{C})$. The character ϖ restricts trivially to the semisimple subgroup $S\mathbf{U} \subset \mathbf{U}$. Thus, by Lemma 1.3.1(i), it remains to show the triviality of its restriction to $\mathbf{Z}$. In fact, we will show directly that $\varpi(t) = 1$ for any $t = \begin{pmatrix} a^{-1} & 0 \\ 0 & \hat{a} \end{pmatrix} \in \mathbf{T}(\mathbb{Q})$, where $a \in (\mathrm{End}(A) \otimes \mathbb{Q})^*$. Note that this implies that $\varpi|_{\mathbf{T}} = 1$ since $\mathbf{T}(\mathbb{Q})$ is dense in $\mathbf{T}$. It suffices to consider the case when $a \in \mathrm{End}(A)$. Then the correspondence $L(t) \subset X_A \times X_A$ is the image of the embedding

$$A \times \hat{A} \to X_A \times X_A : (x, \xi) \mapsto (ax, \xi, x, \hat{a}(\xi)).$$

Hence, in this case $q(t) = \deg(a)$ and $q(t^{-1}) = \deg(\hat{a}) = \deg(a)$, and our assertion follows from (2.5.5). □

Corollary 2.5.6. *For $\widetilde{g} = (g, f_g) \in \mathrm{Spin}(\mathbb{C})$, $\omega \in D_A$ and $x \in \mathcal{N}(A) \otimes \mathbb{C}$ one has*

$$\chi(\ell(\omega), \hat{\rho}(\widetilde{g})^{-1}(x)) = f_g(\omega) \cdot \chi(\ell(g(\omega)), x).$$

Proof. Indeed, we have

$$\chi(\ell(\omega), \hat{\rho}(\widetilde{g})^{-1}(x)) = \chi(\hat{\rho}(\widetilde{g})(\ell(\omega)), x) = f_g(\omega) \cdot \chi(\ell(g(\omega)), x).$$

□

Corollary 2.5.7. *For any $g \in \mathbf{U}(\mathbb{Q})$ one has $q(g) = q(g^{-1})$.*

Examples 2.5.8. 1. If A is an abelian variety of dimension n over $\mathbb{C}$ without complex multiplication then we have $\mathrm{NS}(A) = \mathbb{Z} \cdot H$, where H is an ample generator, and so $\tau \mapsto \tau\phi_H$ gives an identification $\mathfrak{H} \to D_A$, where $\mathfrak{H}$ is the upper half-plane. The group $\mathbf{U}(\mathbb{R})$ can be identified with $\mathrm{SL}(2, \mathbb{R})$ with the action on $\mathfrak{H} \simeq D_A$ given by fractional-linear transformations (1.4.3). Since $\Delta(g)(\tau \cdot \phi_H) = (a + b\tau)^{2n}$, we have a natural splitting

$$\mathrm{SL}(2, \mathbb{R}) \to \mathrm{Spin}_{X_A}(\mathbb{R}) : g \mapsto (g, (a + b\tau)^n).$$

Furthermore, if A is generic then $\mathcal{N}(A) \otimes \mathbb{Q}$ can be identified with the $g+1$-dimensional subspace in $H^*(A, \mathbb{Q})$ spanned by the classes H^i, $i = 0, \dots, g$, and formula (2.5.1) shows that $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathcal{N}(A) \otimes \mathbb{R}$ as on the standard $(g+1)$-dimensional irreducible representation. Assume in addition that ϕ_H is a principal polarization of A. Then we can index simple semihomogeneous vector bundles by rational numbers. Namely, for coprime integers (r, d) with $r > 0$ we set

$$V_{r,d} = V_{\frac{d}{r}\phi_H}.$$

From formula (2.1.9) we get in this case $\operatorname{rk} V_{r,d} = r^n$ (see also [20, Rem. 7.13], [27, ch. 12, exer. 2]). Hence, by Lemma 2.5.2,

$$\operatorname{ch}(V_{r,d}) = \sum_{i=0}^{n} r^{n-i} d^i \cdot \frac{H^i}{i!} \in H^*(A, \mathbb{Z}).$$

Note that for $r = 0$, $d = 1$ this formula gives $\operatorname{ch}(\mathcal{O}_x)$. Using Hirzebruch-Riemann-Roch formula we get the following relations for the form χ on $\mathcal{N}(A) \otimes \mathbb{Q}$:

$$\chi(H^i, H^{n-i}) = (-1)^i n!,$$
$$\chi(\ell(\tau\phi_H), [V_{r,d}]) = (d - r\tau)^n.$$

2. Continuing the previous example assume in addition that $n = \dim A = 3$ (keeping the assumptions that A is principally polarized and generic). Then A is the Jacobian of a curve, so $H^2/2$ is an algebraic class. We claim that the image of the Chern character $\operatorname{ch} : K_0(A) \to H^*(A, \mathbb{Q})$ contains the $\mathbb{Z}$-submodule $K \subset H^*(A, \mathbb{Q})$ spanned by $(H^i/i!)_{0 \le i \le n}$. Indeed, the Chern characters of the structure sheaves of a point and of the curve span the submodule $\mathbb{Z}H^2/2 + \mathbb{Z}H^3/6$. Together with $\operatorname{ch}(\mathcal{O}_A) = 1$ and $\operatorname{ch}(\mathcal{O}(H)) = \exp(H)$ these classes span the whole $\mathbb{Z}$-submodule K. On the other hand, using the above formula we see that for $n \ge 3$ the images of the Chern characters of LI-sheaves (which are all $\mathbf{H}$-equivalent to either $V_{d,r}$ or to $\mathcal{O}_x$) span a proper $\mathbb{Z}$-submodule in K. In particular, the LI-objects do not generate $D^b(A)$ in this case.

3. If A is an elliptic curve over $\mathbb{C}$ with complex multiplication then we have an isomorphism

$$\mathbf{U}(\mathbb{R}) \simeq \mathrm{SL}(2, \mathbb{R}) \times \mathbf{U}(1)/\{\pm 1\},$$

where $\{\pm 1\}$ is embedded into the product diagonally. Also, $D_A = \mathfrak{H}$, the upper half-plane, and $\mathbf{U}(\mathbb{R})$ acts on D_A through the projection to $\mathrm{SL}(2, \mathbb{R})/\{\pm 1\}$. The spin-covering $\mathrm{Spin}_{X_A}(\mathbb{R}) \to \mathbf{U}(\mathbb{R})$ in this case can be identified with the natural covering

$$\mathrm{SL}(2, \mathbb{R}) \times \mathbf{U}(1) \to \mathbf{U}(\mathbb{R}).$$

3. Action on stability spaces

3.1. **Induced t-structures and stabilities.** We refer to [8] for notions related to stability conditions on triangulated categories. All t-structures considered below are assumed to be bounded and nondegenerate (see [3]). All stabilities are assumed to be locally finite and numerical.

We say that a t-structure (resp., a slicing or a stability condition) on $D^b(A)$ is $\mathbf{H}$-invariant, if it is invariant under any functor $T_{(x,\xi)}$ with $(x, \xi) \in A \times \hat{A}$ (see (2.1.3)), i.e., under translations and tensoring by $\operatorname{Pic}^0(A)$. Note that by [29, Cor. 3.5.2], every *full* stability condition is $\mathbf{H}$-invariant.

The general construction of the induced t-structures (resp., stability conditions) from [29] and [18] specializes to the following result on inducing $\mathbf{H}$-invariant t-structures.

Proposition 3.1.1. *Let A and B be abelian varieties of the same dimension, and let $\Phi : D^b(A) \to D^b(B)$ be the LI-functor associated with a Lagrangian correspondence (L, α) from X_A to X_B such that the projections $L \to X_A$ and $L \to X_B$ are surjective, with the right adjoint functor $\Phi' : D^b(B) \to D^b(A)$. Also, let $(D^{\le 0}, D^{\ge 0})$ be an $\mathbf{H}$-invariant*

t-structure on $D^b(A)$. Then there is a unique $\mathbf{H}$*-invariant t-structure* $({}^{\Phi}D^{\le 0}, {}^{\Phi}D^{\ge 0})$ *on* $D^b(B)$*, such that*

$$\Phi(D^{[a,b]}) \subset {}^{\Phi}D^{[a,b]} \tag{3.1.1}$$

It is given by

$${}^{\Phi}D^{[a,b]} = \{F \in D^b(B) \mid \Phi'(F) \in D^{[a,b]}\}. \tag{3.1.2}$$

Similarly, if $(P(t))_{t\in\mathbb{R}}$ is a $\mathbf{H}$*-invariant slicing on $D^b(A)$ then there is a unique* $\mathbf{H}$*-invariant slicing $({}^{\Phi}P(t))_{t\in\mathbb{R}}$ on $D^b(B)$ such that $\Phi(P(t)) \subset {}^{\Phi}P(t)$ for any $t \in \mathbb{R}$. We have ${}^{\Phi}P(t) = (\Phi')^{-1}(P(t))$.*

Proof. First, we observe that by Proposition [31, Prop. 3.2.7], Φ' differs by a shift from the LI-functor associated with the transposed correspondence $(\sigma(L), \alpha^{-1})$, where $\sigma : X_A \times X_B \to X_B \times X_A$ is the permutation of factors. Hence, the same argument as in [31, Lem. 3.3.3] shows that both compositions $\Phi' \circ \Phi$ and $\Phi \circ \Phi'$ are obtained by consecutive extensions from functors $T_{(x,\xi)}$, one of which is the identity functor.

The fact that (3.1.2) defines a t-structure follows from [29, Thm. 2.1.2] once we check that in our situation $\Phi' \circ \Phi$ is t-exact with respect to the original t-structure and $(\Phi \circ \Phi')(F) = 0$ implies $F = 0$. Indeed, the former follows from $\mathbf{H}$-invariance of our t-structure. To check the latter property it suffices to consider the case when F is a coherent sheaf. We observe that the right adjoint functor to $\Phi \circ \Phi'$ sends a structure sheaf of a point $\mathcal{O}_x$ to a sheaf K_x supported on a finite number of points including x. Hence, if $(\Phi \circ \Phi')(F) = 0$ then $\mathrm{Hom}(F, K_x) = 0$ for all $x \in B$, which implies that $F = 0$.

The inclusion (3.1.1) follows from the $\mathbf{H}$-invariance of the original t-structure and from the form of $\Phi' \circ \Phi$. The fact that the new t-structure is $\mathbf{H}$-invariant follows from the $\mathbf{H}$-intertwining property of LI-functors (see (2.1.5) and [31, Lem. 3.2.4]). Now suppose $({}^{\Phi}D_1^{\le 0}, {}^{\Phi}D_1^{\ge 0})$ is another $\mathbf{H}$-invariant t-structure on $D^b(B)$ such that $\Phi(D^{[a,b]}) \subset {}^{\Phi}D_1^{[a,b]}$. Then applying [29, Thm. 2.1.2] again we deduce that

$$D_1^{[a,b]} = \{F \in D^b(A) \mid \Phi(F) \in {}^{\Phi}D_1^{[a,b]}\}$$

is a t-structure on $D^b(A)$ such that $D^{[a,b]} \subset D_1^{[a,b]}$. Hence, $D_1^{[a,b]} = D^{[a,b]}$ and we can rewrite (3.1.2) as

$${}^{\Phi}D^{[a,b]} = \{F \in D^b(B) \mid \Phi\Phi'(F) \in {}^{\Phi}D_1^{[a,b]}\},$$

which implies that ${}^{\Phi}D_1^{[a,b]} \subset {}^{\Phi}D^{[a,b]}$, so these t-structures are the same.

The result about slicings is proved analogously. □

Let $\mathrm{Stab}^{\mathbf{H}}(A)$ denote the space of $\mathbf{H}$-invariant stability conditions on A (it is known to be nonempty for $\dim A \le 2$).

Definition 3.1.2. For $g \in \mathbf{U}(\mathbb{Q})$ and a stability $\sigma = (P(\cdot), Z) \in \mathrm{Stab}^{\mathbf{H}}(A)$ we set

$$g(\sigma) = ({}^{\Phi_g}P(\cdot), Z \circ \hat{\rho}(g)^{-1}),$$

where $\hat{\rho}(g)$ is given by (2.1.17). By Proposition 3.1.1, this defines an action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $\mathrm{Stab}^{\mathbf{H}}(A)$, such that the central element $1 \in \mathbb{Z} \subset \widetilde{\mathbf{U}(\mathbb{Q})}$ sends $(P(\cdot), Z)$ to $(P(\cdot)[1], -Z)$.

The restriction of the above action to the preimage of $\mathbf{U}(\mathbb{Z}) \subset \mathbf{U}(\mathbb{Q})$ is given by the standard action of the autoequivalence group of $D^b(A)$ on $\mathrm{Stab}(A)$ (see [8]).

Proposition 3.1.3. *For every $\widetilde{g} \in \widetilde{\mathbf{U}(\mathbb{Q})}$ the corresponding transformation of $\mathrm{Stab}^{\mathbf{H}}(A)$ is an isometry with respect to the generalized metric $d(\cdot, \cdot)$ introduced in* [8, Prop. 8.1].

Proof. Note that the functor Φ_g sends Harder-Narasimhan constituents of E with respect to σ to those of $\Phi_g(E)$ with respect to $g(\sigma)$, and $Z(\hat{\rho}(g)^{-1}(\Phi_g(E)))$ is a constant multiple (depending only on g) of $Z(E)$. Hence, $d(\sigma_1, \sigma_2) \le d(g(\sigma_1), g(\sigma_2))$. Applying the same inequality to g^{-1} and the pair $(g(\sigma_1), g(\sigma_2))$ we deduce that it is in fact an equality. $\square$

Proposition 3.1.4. *Any LI-object in $D^b(A)$ is semistable with respect to any full stability.*

Proof. Let E be an (L, α)-invariant object in $D^b(A)$, where (L, α) is a Lagrangian pair (with $L \subset X_A$), and let $\sigma = (P(\cdot), Z)$ be a full stability. We can assume that Z takes values in $\mathbb{Q} + i\mathbb{Q} \subset \mathbb{C}$. Indeed, the set of such stabilities is dense in the connected component containing σ, and the semistability of E is a closed condition on σ. Then for a dense set of real numbers t (namely, those with $\tan(\pi t) \in \mathbb{Q}$) the abelian category $P((t, t+1])$ is Noetherian (see [1, Prop. 5.0.1]). Applying the construction of [29] we obtain for each such t the associated *constant family* of t-structures over any base S, which is a certain t-structure on $D^b(A \times S)$, local over S and such that its heart contains the pull-back of $P((t, t+1])$ with respect to the projection $p_1 : A \times S \to A$. Let us take as a base $S = L$ and consider the functor

$$T_{(L,\alpha)} : D^b(A) \to D^b(A \times L)$$

that associates with $F \in D^b(A)$ the natural family of objects $\mathcal{F}$ on $L \times A$ such that the restriction of $\mathcal{F}$ to $\{l\} \times A$ is $\alpha_l \otimes T_l(F)$ for $l \in L$ ($\mathcal{F}$ is obtained from F by taking the pull-back with respect to the map $L \times A \to A : (l, x) \mapsto p_A(l) + x$ and then tensoring the result with a certain line bundle). Since our stability is $\mathbf{H}$-invariant (by [29, Cor. 3.5.2]), this functor is easily seen to be t-exact, i.e., it sends $P((t, t+1])$ to the heart of the corresponding constant t-structure on $D^b(A \times L)$. By definition, (L, α)-invariance structure on E is an isomorphism

$$T_{(L,\alpha)}(E) \simeq p_1^* E.$$

Since both sides are t-exact functors of E, we deduce that the truncations of E with respect to our t-structure are still (L, α)-invariant. Applying this for an appropriate set of phases t we derive that all Harder-Narasimhan constituents of E are (L, α)-invariant. Let E_0 be one of them. Suppose E_0 has cohomological range $[a, b]$ with respect to the standard t-structure. Then $H^b E_0$ and $H^a E_0$ are still (L, α)-invariant, so we have a nonzero morphism $H^b E_0 \to H^a E_0$ (see [31, Thm. 2.4.5]), which gives rise to a nonzero morphism

$$E_0[b] \to H^b E_0 \to H^a E_0 \to E_0[a].$$

By semistability of E_0 we should have $b \le a$, i.e., E_0 is cohomologically pure. Since E_0 is a direct sum of several copies of the generator $S_{L,\alpha}$, it follows that $S_{L,\alpha}$ is also semistable. $\square$

3.2. **$\mathbb{Z}$-covering of $\mathbf{LG}(\mathbb{R})$.** Recall that the action of $\mathbf{U}(\mathbb{R})$ on $\mathbf{LG}(\mathbb{R})$ is transitive (see Prop. 1.4.3), so we have an identification

$$\mathbf{LG}(\mathbb{R}) \simeq \mathbf{U}(\mathbb{R})/\mathbf{P}^{-}(\mathbb{R}). \tag{3.2.1}$$

We have a natural lifting of $\mathbf{P}^{-}(\mathbb{R})$ to a closed subgroup of U^{Δ} (see Lemma 2.3.3). Therefore, the homogeneous space $U^{\Delta}/\mathbf{P}^{-}(\mathbb{R})$ is a $\mathbb{Z}$-covering of $\mathbf{LG}(\mathbb{R})$. Below we will describe this $\mathbb{Z}$-covering explicitly using the homogeneous coordinates $(x : y)$ on $\mathbf{LG}(\mathbb{R})$ (see Sec. 1.4).

Namely, with every $L = (x : y) \in \mathbf{LG}(\mathbb{R})$ we associate a holomorphic function on D_A, defined up to rescaling by a positive constant,

$$\delta(L)(\omega) = \deg(\hat{y} - \hat{x}\omega) = \deg(\omega x - y) \operatorname{mod} \mathbb{R}_{>0},$$

where $\omega \in D_A$. Note that if we change $(x : y)$ to $(x\alpha : y\alpha)$ then this function gets multiplied by $\deg(\alpha) \in \mathbb{R}_{>0}$. It is easy to see that for $g \in \mathbf{U}(\mathbb{R})$ one has

$$\delta(g(0 : \phi_0)) = \Delta(g^{-1}) \operatorname{mod} \mathbb{R}_{>0}, \tag{3.2.2}$$

where $\phi_0 : A \to \hat{A}$ is a polarization and Δ is the 1-cocycle of $\mathbf{U}(\mathbb{R})$ with values in $\mathcal{O}^*(D_A)$ defined in Section 2.3. In particular, $\delta(L)(\omega) \neq 0$ for all $\omega \in D_A$.

Lemma 3.2.1. *For $L \in \mathbf{LG}(\mathbb{R})$ and $g \in \mathbf{U}(\mathbb{R})$ one has*

$$\delta(gL)(g(\omega)) = \delta(L)(\omega) \cdot \Delta(g^{-1})(g(\omega)) = \delta(L)(\omega) \cdot \Delta(g)(\omega)^{-1}.$$

Proof. Pick $g' \in \mathbf{U}(\mathbb{R})$ such that $L = g'(0 : \phi_0)$. Then use (3.2.2) and the cocycle condition for Δ. $\square$

Note also that if we have a Lagrangian subvariety $L \subset A \times \hat{A}$ then viewing L as a point in $\mathbf{LG}(\mathbb{Q})$ we have

$$\delta(L)(\omega) = \deg(\omega p_1 - p_2) \operatorname{mod} \mathbb{R}_{>0}, \tag{3.2.3}$$

where $p_1 : L \to A$ and $p_2 : L \to \hat{A}$ are the projections, and we use the polynomial function $\deg : \operatorname{Hom}(L, \hat{A}) \otimes \mathbb{C} \to \mathbb{C}$.

Definition 3.2.2. We define the $\mathbb{Z}$-covering $p : \widetilde{\mathbf{LG}(\mathbb{R})} \to \mathbf{LG}(\mathbb{R})$ by setting

$$\widetilde{\mathbf{LG}(\mathbb{R})} = \{(L, f) \in \mathbf{LG}(\mathbb{R}) \times (\mathcal{O}(D_A)/i\mathbb{R}) \mid \delta(L) = \exp(2\pi i f) \operatorname{mod} \mathbb{R}_{>0}\}.$$

We also set

$$\widetilde{\mathbf{LG}(\mathbb{Q})} := p^{-1}(\mathbf{LG}(\mathbb{Q})) \subset \widetilde{\mathbf{LG}(\mathbb{R})}.$$

When we need to stress the dependence on A we write $\widetilde{\mathbf{LG}_A(\mathbb{R})}$ (resp., $\widetilde{\mathbf{LG}_A(\mathbb{Q})}$). We have an action of U^{Δ} on $\widetilde{\mathbf{LG}(\mathbb{R})}$ given by

$$(g, f_g) \cdot (L, f_L) = (gL, f_L(g^{-1}(\omega)) + f_g(g^{-1}(\omega))).$$

The fact that this action is well defined follows from Lemma 3.2.1.

Proposition 3.2.3. *(i) There exists a unique bijection*

$$\overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^* \to \widetilde{\mathbf{LG}(\mathbb{Q})} : F \mapsto \widetilde{L}_F, \tag{3.2.4}$$

lifting the natural projection $F \mapsto L_F$ to $\mathbf{LG}(\mathbb{Q})$, sending $\mathcal{O}_x$ to $((0:\phi_0),0) \in \widetilde{\mathbf{LG}(\mathbb{Q})}$ (where $\phi_0 : A \to \hat{A}$ is a polarization), and $\widetilde{\mathbf{U}(\mathbb{Q})}$-equivariant, where the action on $\widetilde{\mathbf{LG}(\mathbb{Q})}$ is induced by the embedding $\iota : \widetilde{\mathbf{LG}(\mathbb{Q})} \to U^{\Delta}$.
(ii) Let V_ϕ be the semihomogeneous vector bundle associated with $\phi \in \mathrm{NS}(A)\otimes\mathbb{Q}$, so that $L_{V_\phi} = \Gamma(\phi)$ (see (2.1.4)). Then

$$\widetilde{L}_{V_\phi} = (\Gamma(\phi), (2\pi i)^{-1}\cdot \log(\deg(\omega-\phi)) \bmod i\mathbb{R}), \tag{3.2.5}$$

where the branch of $\log(\deg(\cdot))$ *is normalized by* $\operatorname{Im}\log(\deg(iH)) = \operatorname{Arg}(\deg(iH)) = -g\pi$ *for ample H.*

Proof. (i) First, let us compute the stabilizer subgroup $\mathrm{St} \subset \widetilde{\mathbf{U}(\mathbb{Q})}$ of the class of $\mathcal{O}_x$ in $\overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^*$. By considering the action on the corresponding Lagrangian we see that St is a certain lifting of $\mathbf{P}^-(\mathbb{Q}) \subset \mathbf{U}(\mathbb{Q})$ to $\widetilde{\mathbf{U}(\mathbb{Q})}$. From the explicit form of the functors Φ_t for $t \in \mathbf{T}(\mathbb{Q})$ (see Prop. 2.2.1(ii)) we see that these functors preserve $\mathcal{O}_x$ up to $\mathbf{H}$-equivalence and $\mathbb{N}^*$. Therefore, St is the lifting of $\mathbf{P}^-(\mathbb{Q})$ described in Corollary 2.2.2. By Lemma 2.3.3, $\iota(\mathrm{St})$ is exactly the stabilizer of the point $((0:\phi_0),0) \in \widetilde{\mathbf{LG}(\mathbb{Q})}$. Hence, there is a well-defined $\widetilde{LG(\mathbb{Q})}$-equivariant map (3.2.4). Since this is a map of $\mathbb{Z}$-torsors over $\mathbf{LG}(\mathbb{Q})$ (see Prop. 2.1.2), it is a bijection.
(ii) Assume first that ϕ is non-degenerate, i.e., $\phi \in \mathrm{NS}^0(A,\mathbb{Q})$. Consider the element $g^+_{\phi^{-1}} \in \mathbf{N}^+(\mathbb{Q})$ as in Proposition 2.2.1(i). Then

$$g^+_{\phi^{-1}}(0:\phi_0) = (\phi^{-1}\phi_0 : \phi_0) = (1:\phi) = \Gamma(\phi).$$

By Proposition 2.2.1(i), under the canonical lifting of $\mathbf{N}^+(\mathbb{Q})$ to $\widetilde{\mathbf{U}(\mathbb{Q})}$ the lifting of $g^+_{\phi^{-1}}$ corresponds to the kernel $S(g^+_{\phi^{-1}})[i(\phi)]$ (note that $i(\phi^{-1}) = i(\phi)$). On the other hand, its canonical lifting to U^{Δ} is

$$\widetilde{g} = (g^+_{\phi^{-1}}, -\log(\deg(1+\phi^{-1}\omega))/2\pi i),$$

where we use the branch of $\operatorname{Arg}\deg(1+\phi^{-1}\omega)$ that tends to 0 as $\omega \to 0$ (see Lemma 2.3.4). By the $\widetilde{\mathbf{U}(\mathbb{Q})}$-equivariance of the map (3.2.4), we obtain that the object $V_\phi[i(\phi)]$ is mapped under this map to

$$\widetilde{g}\cdot((0:\phi_0),0) = (\Gamma(\phi), \log(\deg(1-\phi^{-1}\omega))/2\pi i \bmod i\mathbb{R}) =$$
$$(\Gamma(\phi), (2\pi i)^{-1}\cdot\log(\deg(\omega-\phi)) \bmod i\mathbb{R})$$

with the same choice of the argument as above. Recall that if we choose the branch of $\operatorname{Arg}\deg(\omega-\phi)$ in such a way that $\operatorname{Arg}\deg(inH-\phi)$ will be πg then we will obtain

$$\operatorname{Arg}\deg(-\phi) = 2\pi i(-\phi) = 2\pi(g - i(\phi))$$

(see Corollary 1.2.2). Subtracting $2\pi g$ we get the branch that gives the limit $-2\pi i(\phi)$ as $\omega \to 0$ which is exactly what we get for the image of V_ϕ. This proves the required

statement in the case when $\phi \in \mathrm{NS}^0(A,\mathbb{Q})$. The general case follows by using the action of the subgroup $\mathbf{N}^-(\mathbb{Q}) \subset \widetilde{\mathbf{U}(\mathbb{Q})}$ (see Ex. 2.1.3). Indeed, this action changes both sides (3.2.5) by adding to ϕ an arbitrary element of $\mathrm{NS}(A) \otimes \mathbb{Q}$. $\square$

Recall that we can view $\mathrm{NS}(A) \otimes \mathbb{R}$ as an open subset of $\mathbf{LG}(\mathbb{R})$ via the map $\phi \mapsto (1 : \phi) = \Gamma(\phi)$. Part (ii) of the above Proposition implies that we have a commutative diagram

$$\begin{tikzcd} \mathrm{NS}(A) \otimes \mathbb{Q} \arrow[r, "\phi \mapsto V_\phi"] \arrow[d] & \overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^* \arrow[d] \\ \mathrm{NS}(A) \otimes \mathbb{R} \arrow[r] & \widetilde{\mathbf{LG}(\mathbb{R})} \end{tikzcd}$$

where the right vertical arrow is (3.2.4) and the bottom arrow is the continuous section of the projection $\widetilde{\mathbf{LG}(\mathbb{R})} \to \mathbf{LG}(\mathbb{R})$ over $\mathrm{NS}(A) \otimes \mathbb{R}$ given by

$$\mathrm{NS}(A) \otimes \mathbb{R} \to \widetilde{\mathbf{LG}(\mathbb{R})} : \phi \mapsto (\Gamma(\phi), f_\phi), \tag{3.2.6}$$

where $f_\phi \in \mathcal{O}(D_A) \bmod i\mathbb{R}$ is the branch of $(2\pi i)^{-1} \cdot \log(\deg(\omega - \phi)) \bmod i\mathbb{R}$ satisfying

$$\lim_{n\to\infty} f_\phi(inH) = -g/2$$

for any ample H.

Definition 3.2.4. We define the double covering $p^{\mathrm{spin}} : \mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R}) \to \mathbf{LG}_A(\mathbb{R})$ by setting $\mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R}) = \widetilde{\mathbf{LG}(\mathbb{R})}/2\mathbb{Z}$. Explicitly,

$$\mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R}) = \{(L,\varphi) \in \mathbf{LG}(\mathbb{R}) \times (\mathcal{O}(D_A)/\mathbb{R}_{>0}) \mid \delta(L) = \varphi^2 \bmod \mathbb{R}_{>0}\}.$$

We also set $LG^{\mathrm{spin}}(A,\mathbb{Q}) = (p^{\mathrm{spin}})^{-1}(\mathbf{LG}(\mathbb{Q}))$.

The isomorphism $\mathrm{Spin}(\mathbb{R}) \simeq U^\Delta/2\mathbb{Z}$ induces a transitive action of $\mathrm{Spin}(\mathbb{R})$ on $\mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R})$ (and of $\mathbf{U}(\mathbb{Q})^{\mathrm{spin}}$ on $\mathbf{LG}^{\mathrm{spin}}(A,\mathbb{Q})$).

We also have a natural $\widetilde{\mathbf{U}(\mathbb{Q})}$-equivariant map

$$\overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^* \to \mathcal{N}(A) \otimes \mathbb{Q}/\mathbb{Q}_{>0} : F \mapsto [F] \bmod \mathbb{Q}_{>0}$$

(see (2.4.8)), which we can view as a map from $\widetilde{\mathbf{LG}(\mathbb{Q})}$ using the bijection (3.2.4). The equivariance of this map with respect to the $\mathbb{Z}$-action implies that it factors through $\mathbf{LG}^{\mathrm{spin}}(A,\mathbb{Q})$. Furthermore, we claim that it extends to a continuous $\mathrm{Spin}(\mathbb{R})$-equivariant map

$$\mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R}) \to \mathcal{N}(A) \otimes \mathbb{R}/\mathbb{R}_{>0} \tag{3.2.7}$$

such that we have a commutative diagram:

$$\begin{tikzcd}
\overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^* \arrow[r] \arrow[d] & \mathcal{N}(A) \otimes \mathbb{Q}/\mathbb{Q}_{>0} \arrow[d] \\
\mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R}) \arrow[r] & \mathcal{N}(A) \otimes \mathbb{R}/\mathbb{R}_{>0}
\end{tikzcd}$$

Indeed, we can define (3.2.7) by sending $\widetilde{g}((0:\phi_0),1)$ to $\widetilde{g}[\mathcal{O}_x] \bmod \mathbb{R}_{>0}$ for $\widetilde{g} \in \mathrm{Spin}(\mathbb{R})$. To check that this map is well defined we observe that $\mathbb{Q}$-points $\mathbf{P}^-(\mathbb{Q})$ are dense (with respect to the classical topology) in the stabilizer $\mathbf{P}^-(\mathbb{R})$ of the point $((0:\phi_0),1) \in \mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R})$. Since $\mathbf{P}^-(\mathbb{Q}) \subset \mathbf{U}(\mathbb{Q})^{\mathrm{spin}}$ leaves the class $[\mathcal{O}_x] \in S\mathcal{N}(A) \otimes \mathbb{R}$ invariant, this proves our claim.

Lemma 3.2.5. *The section* (3.2.6) *induces a section*

$$\mathrm{NS}(A) \otimes \mathbb{R} \to \mathbf{LG}^{\mathrm{spin}}(A,\mathbb{R}) \tag{3.2.8}$$

which sends $\phi \in \mathrm{NS}(A) \otimes \mathbb{R}$ *to* $(\Gamma(\phi), \chi(\phi - \omega) \bmod \mathbb{R}_{>0})$.

Proof. Since $\chi(\phi-\omega)^2 = \deg(\phi-\omega) = \deg(\omega-\phi)$, this follows from the fact that the argument of $\chi(\phi - inH)$ tends to $-g\pi/2 \bmod 2\pi\mathbb{Z}$ as $n \to \infty$. $\square$

Example 3.2.6. In the case when $A = E^n$, where E is an elliptic curve without complex multiplication we can identify $\mathrm{End}(A)$ with the algebra of $n \times n$-matrices over $\mathbb{Z}$, and $\mathrm{NS}(A)$ with symmetric matrices. Note that for $M \in \mathrm{End}(A)$ we have $\deg(M) = \det(M)^2$ and for $\phi \in \mathrm{NS}(A)$ we have $\chi(\phi) = \det(\phi)$. In a coordinate-free notation, if $A = E \otimes \Lambda$, where Λ is a free $\mathbb{Z}$-module of rank n, then elements of $\mathrm{NS}(A)$ can be viewed as $\mathbb{Z}$-valued symmetric bilinear forms on Λ, and the function χ is given by the discriminant. The group $\mathbf{U}$ in this case is the symplectic group Sp_{2n} and the variety $\mathbf{LG}_A$ is the Lagrangian Grassmannian associated with the $2n$-dimensional symplectic vector space. Also, D_A is the Siegel upper half-plane $\mathfrak{H}_n$ and the covering $U^\Delta \to \mathrm{Sp}_{2n}$ corresponds to a choice of argument of $Z \mapsto \det(A+BZ)^2$, where $Z \in \mathfrak{H}_n$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$. Thus, U^Δ contains the universal covering $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ of $\mathrm{Sp}(2n,\mathbb{R})$ as a subgroup of index 2 (cf. [23, Ex. 4.15]). Now let us consider our lifting of $\mathbf{P}^-(\mathbb{R})$ to U^Δ. It is easy to check that the restriction of the projection to $U^\Delta/\widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \simeq \{\pm 1\}$ to $\mathrm{GL}(n,\mathbb{R}) \subset \mathbf{P}^-(\mathbb{R})$ can be identified with the homomorphism $A \mapsto \operatorname{sign}\det(A)$. It follows that $U^\Delta = \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \cdot \mathbf{P}^-(\mathbb{R})$, and $\mathbf{P}^-(\mathbb{R}) \cap \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is the semidirect product of $\mathbf{N}^-(\mathbb{R})$ and of $\mathrm{GL}^+(n,\mathbb{R})$ (matrices with positive determinant). Hence, we can identify $\widetilde{\mathbf{LG}_A(\mathbb{R})}$ with the quotient of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ by a connected subgroup, so $\widetilde{\mathbf{LG}_A(\mathbb{R})}$ is simply connected. It follows that in this case $\widetilde{\mathbf{LG}_A(\mathbb{R})}$ is the universal covering of the Lagrangian Grassmannian $\mathbf{LG}_A(\mathbb{R})$.

3.3. **Phase function.** Since $\Delta : \mathbf{U}(\mathbb{R}) \to \mathcal{O}^*(D_A)$ is a 1-cocycle (see Lemma 2.3.1), it defines a natural action of the group $\mathbf{U}(\mathbb{R})$ (by holomorphic automorphisms) on the trivial $\mathbb{C}^*$-bundle over D_A. We have constructed the central extension $U^\Delta \to \mathbf{U}(\mathbb{R})$ by $\mathbb{Z}$ in such

a way that Δ lifts to a 1-cocycle of U^{Δ} with coefficients in $\mathcal{O}(D_A)$. In other words, we obtain the action of U^{Δ} on $D_A \times \mathbb{C}$ (respecting the structure of a $\mathbb{C}$-space), which we view as a universal covering of $D_A \times \mathbb{C}^*$ (in Sec. 3.4 we will relate this covering to the Bridgeland's stability space in the case $\dim A = 2$). Explicitly, this action is given by

$$(g, f) \cdot (\omega, z) = (g(\omega), z - f(\omega)), \tag{3.3.1}$$

where $(g, f) \in U^{\Delta}$ and $(\omega, z) \in D_A \times \mathbb{C}$.

On the other hand, we have a transitive action of U^{Δ} on the $\mathbb{Z}$-covering $\widetilde{\mathbf{LG}(\mathbb{R})}$ of $\mathbf{LG}(\mathbb{R})$. By definition of this $\mathbb{Z}$-covering, we have a continuous function

$$\mathbf{f}_0 : D_A \times \widetilde{\mathbf{LG}(\mathbb{R})} \to \mathbb{R} : (\omega, (L, f_L)) \mapsto \operatorname{Re} f_L(\omega).$$

We can extend it to a continuous function on $(D_A \times \mathbb{C}) \times \widetilde{\mathbf{LG}(\mathbb{R})}$ setting

$$\mathbf{f}((\omega, z), \widetilde{L}) = \operatorname{Re}(z) + \mathbf{f}_0(\omega, \widetilde{L}),$$

where $\widetilde{L} \in \widetilde{\mathbf{LG}(\mathbb{R})}$.

Lemma 3.3.1. *The function* $\mathbf{f}$ *is* U^{Δ}*-invariant, i.e., for* $\widetilde{g} \in U^{\Delta}$ *and* $(\sigma, \widetilde{L}) \in (D_A \times \mathbb{C}) \times \widetilde{\mathbf{LG}(\mathbb{R})}$ *one has*

$$\mathbf{f}(\widetilde{g}(\sigma), \widetilde{g}(\widetilde{L})) = \mathbf{f}(\sigma, \widetilde{L}).$$

The proof is straightforward.

Now using the map $F \mapsto \widetilde{L}_F$ of Proposition 3.2.3, we define the *phase function*

$$(D_A \times \mathbb{C}) \times \overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^* \to \mathbb{R} : (\sigma, \widetilde{L}) \mapsto \phi_\sigma(F) := \mathbf{f}(\sigma, \widetilde{L}_F), \tag{3.3.2}$$

where $\sigma \in D_A \times \mathbb{C}$. Note that we have

$$\phi_{(\omega,z)}(F) = \phi_{(\omega,0)}(F) + \operatorname{Re}(z). \tag{3.3.3}$$

In Sec. 3.4 we will show that in the surface case the function ϕ_σ gives the phases of LI-objects with respect to the Bridgeland's stability condition on $D^b(A)$ associated with $\sigma \in D_A \times \mathbb{C}$. In the following theorem we check some of the properties of ϕ_σ that conform with the conjecture that the corresponding stability condition exists in the higher-dimensional case as well.

Theorem 3.3.2. *The phase function* $\phi_\sigma(F)$ *satisfies the following properties.*

(i) This function is $\widetilde{\mathbf{U}(\mathbb{Q})}$*-invariant, i.e.,*

$$\phi_{\widetilde{g}(\sigma)}(\widetilde{g}(F)) = \phi_\sigma(F),$$

where the action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ *on* $D_A \times \mathbb{C}$ *is induced by* (3.3.1) *via the homomorphism* $\iota : \widetilde{\mathbf{U}(\mathbb{Q})} \to U^{\Delta}$. *In particular, for* $n \in \mathbb{Z}$,

$$\phi_\sigma(F[n]) = \phi_\sigma(F) + n.$$

(ii) For $\sigma = (\omega, z)$ *and* $F \in \overline{\mathrm{SH}}^{LI}(A)$ *one has*

$$\exp(\pi i z) \cdot \chi(\ell(\omega), [F]) \in \mathbb{R}_{>0} \cdot \exp(\pi i \phi_\sigma(F)), \tag{3.3.4}$$

where $[F] \in \mathcal{N}(A) \otimes \mathbb{R}$ *is the numerical class of* F.

(iii) For a semihomogeneous vector bundle V_ϕ associated with $\phi \in \mathrm{NS}(A) \otimes \mathbb{Q}$ one has

$$\phi_{(\omega,z)}(V_\phi) = \mathrm{Re}(z) + \frac{1}{2\pi}\mathrm{Arg}(\deg(\omega - \phi)),$$

where the branch of $\mathrm{Arg}(\deg(\cdot))$ *is normalized by* $\mathrm{Arg}(\deg(iH)) = -g\pi$ *for ample H.*
(iv) for a pair of LI-objects F_1 and F_2 such that the corresponding Lagrangians L_{F_1} and L_{F_2} in $A \times \hat{A}$ are transversal one has

$$\phi_\sigma(F_1) \le \phi_\sigma(F_2) + i(F_1, F_2), \tag{3.3.5}$$

where $i(F_1, F_2)$ is the index of the pair (F_1, F_2), i.e., the number such that $\mathrm{Ext}^i(F_1, F_2) = 0$ *for $i \ne i(F_1, F_2)$ (it exists by* [31, Cor. 3.2.12]*).*

Proof. (i) The invariance follows from Lemma 3.3.1. The second assertion follows from this:

$$\phi_{(\omega,z)}(F) = \phi_{(1,n)\cdot(\omega,z)}(F[n]) = \phi_{(\omega,z-n)}(F[n]) = \phi_{(\omega,z)}(F[n]) - n,$$

where in the last equality we used (3.3.3).
(ii) By part (i), the right-hand side of (3.3.4) is invariant under the diagonal action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on (σ, F). We claim that the same is true for the left-hand side (modulo $\mathbb{R}_{>0}$). Indeed, by Corollary 2.5.6, for $\widetilde{g} = (g, f_g) \in U^\Delta$ we have

$$\chi(\ell(\omega), [F]) = \exp(-\pi i f_g(\omega)) \cdot \chi(\ell(g(\omega)), [\widetilde{g}(F)]) \bmod \mathbb{R}_{>0}$$

(recall that the map $F \mapsto [F] \bmod \mathbb{N}^*$ is compatible with the projection $U^\Delta \to \mathrm{Spin}(\mathbb{R})$ sending $(g, f_g) \in U^\Delta$ to $(g, \exp(-\pi i f_g))$, see (2.3.3)). This immediately implies that the left-hand side of (3.3.4) is invariant modulo $\mathbb{R}_{>0}$ with respect to the diagonal action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on the pair $(\sigma, F) \in (D_A \times \mathbb{C}) \times \overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^*$.

Thus, it is enough to check the equality for $F = \mathcal{O}_x$. We have $\chi(\ell(\omega), [\mathcal{O}_x]) = 1$ for all ω. On the other hand, by definition of the map of Proposition 3.2.3, $\mathbf{f}_0(\omega, \widetilde{L}_{\mathcal{O}_x}) = 0$, so $\phi_{(\omega,z)}(\mathcal{O}_x) = \mathrm{Re}(z)$.
(iii) This follows from Proposition 3.2.3(ii).
(iv) By $\widetilde{\mathbf{U}(\mathbb{Q})}$-invariance of both parts with respect to the diagonal action on the pair (F_1, F_2), it is enough to consider the case when $F_2 = \mathcal{O}_x$. Note that in this case the transversality assumption implies that $L_{F_1} = \Gamma(\phi)$ for $\phi \in \mathrm{NS}(A)_\mathbb{Q}$, so $F_1 = V_\phi[n]$ for some $n \in \mathbb{Z}$, where V_ϕ is the simple semihomogeneous bundle associated with ϕ. Since $\phi_{(\omega,z)}(\mathcal{O}_x) = \mathrm{Re}(z)$, by part (iii), the required inequality is equivalent to

$$\mathrm{Arg}(\deg(\omega - \phi)) \le 0,$$

where $\mathrm{Arg}(\deg(\cdot))$ is normalized by $\mathrm{Arg}(\deg(iH)) = -g\pi$. But this follows immediately from Lemma 1.2.3(ii). □

Remark 3.3.3. By Lemma 1.2.3(i), for $F_1 = \mathcal{O}$, $F_2 = V_\phi$ and $\sigma = (iH, 0)$, where H is an ample class, the inequality (3.3.5) can be replaced by a stronger one:

$$\phi_{iH,0}(\mathcal{O}) < \phi_{iH,0}(V_\phi) + \frac{i(\phi)}{2}.$$

However, this inequality is not invariant with respect to the group action considered above, so it cannot be extended to the case of arbitrary $\sigma \in D_A \times \mathbb{C}$.

The following property is also motivated by the picture with the stability conditions for $\dim A = 2$ (see Sec. 3.4 below).

Proposition 3.3.4. *The fibers of the map*

$$\mathcal{Z} : D_A \times \mathbb{C} \to \mathrm{Hom}(\mathcal{N}(A), \mathbb{C}) : (\omega, z) \mapsto \exp(\pi i z)\chi(\ell(\omega), \cdot)$$

are exactly the orbits of the action of $2\mathbb{Z} \subset \mathbb{C}$ *by translations on the second factor.*

Proof. Suppose

$$\exp(\pi i z)\chi(\ell(\omega), [F]) = \exp(\pi i z')\chi(\ell(\omega'), [F])$$

for all F. Since $\chi(\ell(\cdot), [\mathcal{O}_x]) = 1$, this implies that $\exp(\pi i z) = \exp(\pi i z')$. Using the action of $2\mathbb{Z}$ we can assume that $z = z'$. Now the fact that $\omega = \omega'$ follows from Corollary 1.1.2. □

Example 3.3.5. Recall that the standard stability condition on an elliptic curve has $Z(F) = -\deg(F) + i\,\mathrm{rk}(F)$ and semistable objects that are shifts of semistable bundles and torsion sheaves. The corresponding phase function ϕ^{st} satisfies

$$\phi^{st}(F) = \phi_{(i,0)}(F) + 1$$

for any semistable F. Indeed, this follows from the formulas

$$\phi^{st}(\mathcal{O}_x) = 1, \quad \phi^{st}(V_{d/r}) = \frac{\mathrm{Arg}(i - d/r)}{\pi},$$

where $V_{d/r}$ is the simple bundle of degree d and rank r and we normalize the argument in the upper half-plane by $\mathrm{Arg}(i) = 1/2$.

3.4. **Stability conditions on abelian surfaces.** In this section, assuming that $\dim A = 2$ we will identify the action of $\iota(\widetilde{\mathbf{U}(\mathbb{Q})}) \subset U^{\Delta}$ on $D_A \times \mathbb{C}$ with the natural action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on the component $\mathrm{Stab}^{\dagger}$ of Bridgeland's stability space $\mathrm{Stab}(A)$ of $D^b(A)$ described in [9, Sec. 15].

Recall that the stability space $\mathrm{Stab}(A)$ carries a natural continuous action of the group $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, the universal cover of $\mathrm{GL}^+(2, \mathbb{R})$, that can be described as the set of pairs (T, f), where $T \in \mathrm{GL}^+(2, \mathbb{R})$ and $f : \mathbb{R} \to R$ is an increasing map with $f(t+1) = f(t)+1$ such that the map induced by T on $\mathbb{R}^2 \setminus \{0\}/\mathbb{R}_{>0} \simeq \mathbb{R}/2\mathbb{Z}$ coincides with $f \bmod 2\mathbb{Z}$. We use the left action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\mathrm{Stab}(A)$: a pair (T, f) maps a stability condition $(Z, \mathcal{P})$ to the stability $(T \circ Z, \mathcal{P}')$, where $\mathcal{P}'(t) = \mathcal{P}(f^{-1}(t))$. Note that $n \mapsto ((-1)^n, t \mapsto t+n)$ gives an embedding $\mathbb{Z} \to \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $2\mathbb{Z}$ is the kernel of the projection to $\mathrm{GL}^+(2, \mathbb{R})$.

Recall that for each $\omega = i\alpha + \beta \in D_A$ Bridgeland defined a stability condition on $D^b(A)$ with the central charge

$$Z_\omega(F) = -\chi(\ell(\omega), [F])$$

and with each $\mathcal{O}_x$ stable of phase 1. This defines a submanifold $V(A) \subset \mathrm{Stab}(A)$, isomorphic to D_A, which is a section of the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on a connected component $\mathrm{Stab}^{\dagger}(A) \subset \mathrm{Stab}(A)$, so that we have an isomorphism

$$V(A) \times \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \simeq \mathrm{Stab}^{\dagger}(A) \tag{3.4.1}$$

(see [9, Sec. 11, 15])[2].

We have a natural embedding $\mathbb{C}^* = \mathrm{GL}(1,\mathbb{C}) \to \mathrm{GL}^+(2,\mathbb{R})$ and the corresponding homomorphism of universal coverings $\mathbb{C} \hookrightarrow \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ (where we use the map $\mathbb{C} \to \mathbb{C}^* : z \mapsto \exp(\pi i z)$). Hence, from the isomorphism (3.4.1) we obtain an embedding

$$D_A \times \mathbb{C} \simeq V(A) \times \mathbb{C} \hookrightarrow \mathrm{Stab}^\dagger(A). \tag{3.4.2}$$

Note that the central charge corresponding to a point $(\omega, z) \in D_A \times \mathbb{C}$ is

$$Z_{(\omega,z)}(F) = -\exp(\pi i z)\chi(\ell(\omega), [F]),$$

and the phase of $\mathcal{O}_x$ with respect to this stability is

$$\phi^{Br}_{(\omega,z)}(\mathcal{O}_x) = 1 + \mathrm{Re}(z) \tag{3.4.3}$$

Recall that the non-empty fibers of the projection

$$\mathcal{Z} : \mathrm{Stab}^\dagger(A) \to \mathrm{Hom}(\mathcal{N}(A), \mathbb{C})$$

are exactly the orbits of the action of $2\mathbb{Z} \subset \mathbb{C} \subset \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ (see [9, Thm. 15.2]). Hence, we have

$$V(A) \times \mathbb{C} = \mathcal{Z}^{-1}(\mathcal{Z}(V(A) \times \mathbb{C})). \tag{3.4.4}$$

The image $\mathcal{Z}(V(A)\times\mathbb{C})$ coincides with $\mathbb{C}^*\cdot\ell(D_A) \subset \mathcal{N}(A)\otimes\mathbb{C}$, where we identify $\mathcal{N}(A)\otimes\mathbb{C}$ with $\mathrm{Hom}(\mathcal{N}(A),\mathbb{C})$ using $\chi(\cdot,\cdot)$.

Recall that we have an action of the group $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $\mathrm{Stab}^{\mathbf{H}}(A)$ defined using functors Φ_g (see Def. 3.1.2). Also, note that by [29, Cor. 3.5.2], we have an inclusion $\mathrm{Stab}^\dagger(A) \subset \mathrm{Stab}^{\mathbf{H}}(A)$ since all stabilities in $\mathrm{Stab}^\dagger(A)$ are full.

Proposition 3.4.1. *The subset $V(A) \times \mathbb{C} \subset \mathrm{Stab}^{\mathbf{H}}(A)$ is invariant with respect to the action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ and the induced action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $V(A)\times\mathbb{C} \simeq D_A\times\mathbb{C}$ is exactly* (3.3.1).

Proof. First, let us look at the action on central charges. Applying Corollary 2.5.6 to the element $\widetilde{g} = (g, \exp(-\pi i f)) \in \mathrm{Spin}(\mathbb{R})$ coming from an element $(g, f) = \iota(g') \in U^\Delta$ where $g' \in \widetilde{\mathbf{U}(\mathbb{Q})}$, we get

$$Z_{(\omega,z)}(\hat{\rho}(g')^{-1}F) = Z_{(\omega,z)}(\hat{\rho}(\widetilde{g})^{-1}[F]) = Z_{\iota(g')\cdot(\omega,z)}(F).$$

(see (3.3.1)). In particular, the transformed central charge is still in $\mathbb{C}^* \cdot \ell(D_A)$. Recall that the connected component $\mathrm{Stab}^\dagger(A)$ is characterized by the condition that the central charge is in the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of $\ell(D_A)$ and $\mathcal{O}_x$ are stable of the same phase for all $x \in A$. Furthermore, by [9, Lem. 12.2], is is enough to require all $\mathcal{O}_x$ to be semistable of the same phase (due to the absence of spherical objects—see [9, Lem. 15.1]). In our case the condition on the central charge is satisfied by the above computation, and the semistability of $\mathcal{O}_x$ follows from Proposition 3.1.4, so we get the inclusion $g'(V(A)\times\mathbb{C}) \subset \mathrm{Stab}^\dagger$. Taking into account (3.4.4) we derive the required inclusion

$$g'(V(A) \times \mathbb{C}) \subset V(A) \times \mathbb{C} \subset \mathrm{Stab}^\dagger.$$

[2]Conjecturally, $\mathrm{Stab}^\dagger(A) = \mathrm{Stab}(A)$.

Furthermore, we obtain that the action of $g' \in \widetilde{\mathbf{U}(\mathbb{Q})}$ on $D_A \times \mathbb{C}$ differs from the action (3.3.1) by the translation by an element in $2\mathbb{Z} \subset \mathbb{C}$. Thus, the difference between the two action is given by a homomorphism $\widetilde{\mathbf{U}(\mathbb{Q})} \to 2\mathbb{Z}$. Note that the element $1 \in \mathbb{Z} \subset \widetilde{\mathbf{U}(\mathbb{Q})}$ acts on a stability in $\mathrm{Stab}(A)$ by changing the central charge Z to $-Z$ and adding -1 to all the phases. Since this matches with its action on $D_A \times \mathbb{C}$ given by (3.3.1), the above homomorphism factors through a homomorphism $\mathbf{U}(\mathbb{Q}) \to 2\mathbb{Z}$. Next, we observe that the action of $\mathbf{P}^-(\mathbb{Q}) \subset \widetilde{\mathbf{U}(\mathbb{Q})}$ preserves the phase of $\mathcal{O}_x$ (see the proof of Prop. 3.2.3(i)). On the other hand, $\iota(\mathbf{P}^-(\mathbb{Q})) \subset U^\Delta$ consists of elements (g, f) with $\mathrm{Re}(f) = 0$ (see Lemma 2.3.3), so taking into account the formula (3.4.3) we deduce that the homomorphism $\mathbf{U}(\mathbb{Q}) \to 2\mathbb{Z}$ is trivial on $\mathbf{P}^-(\mathbb{Q})$. It remains to apply Lemma 1.3.5. $\square$

Corollary 3.4.2. *There is a transitive continuous action of* $U^\Delta \times \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ *on* $\mathrm{Stab}^\dagger(A)$, *extending the action of* $\widetilde{\mathbf{U}(\mathbb{Z})}$ *(coming from autoequivalences of* $D^b(A)$*) and the standard action of* $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$.

Proof. This follows from the identification (3.4.1) and from the transitivity of the action of U^Δ on $D_A \times \mathbb{C}$. Note that our action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $\mathrm{Stab}^{\mathbf{H}}(A)$ extends the standard action of $\widetilde{\mathbf{U}(\mathbb{Z})}$ by autoequivalences of $D^b(A)$. $\square$

Theorem 3.4.3. *For any* $\sigma = (\omega, z) \in D_A \times \mathbb{C}$ *and any LI-object* $F \in D^b(A)$, *let* $\phi_\sigma^{Br}(F)$ *be the phase of* F *with respect to the corresponding Bridgeland's stability condition. Then*

$$\phi_\sigma^{Br}(F) = \phi_\sigma(F) + 1,$$

where the function ϕ_σ *is given by* (3.3.2).

Proof. The assertion is true for $F = \mathcal{O}_x$. Also Theorem 3.3.2(i) together with Proposition 3.4.1 imply that both sides are invariant with respect to the action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on the pair (σ, F). It remains to use transitivity of the action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $\overline{\mathrm{SH}}^{LI}(A)/\mathbb{N}^*$. $\square$

3.5. **Mirror symmetry and phases.** In the case when $A = E^n$, where E is an elliptic curve without complex mutliplication, we can interpret the phase function of Sec. 3.3 in terms of the Fukaya category of the mirror dual abelian variety.

Let $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be an elliptic curve over $\mathbb{C}$ and let Λ be a free $\mathbb{Z}$-module of rank n. We set

$$A = \Lambda \otimes \mathbb{C}/(\Lambda \otimes (\mathbb{Z} + \tau\mathbb{Z})) \simeq E^n,$$

so that we have a natural isomorphism

$$\Gamma_A := H_1(A, \mathbb{Z}) \simeq \Lambda \oplus \Lambda,$$

where the second summand corresponds to $\Lambda \otimes \tau$. The natural polarization of E given by the hermitian form $H_\tau(z_1, z_2) = \frac{z_1 \overline{z_2}}{\mathrm{Im}\,\tau}$ induces an isomorphism

$$\hat{A} \simeq \Lambda^* \otimes \mathbb{C}/(\Lambda^* \otimes (\mathbb{Z} + \tau\mathbb{Z}),$$

where $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$.

Assuming that E has no complex multiplication we obtain identifications $\mathrm{End}(A) \simeq \mathrm{End}_{\mathbb{Z}}(\Lambda)$, $\mathrm{Hom}(A, \hat{A}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda^*)$, and $\mathrm{NS}(A) \simeq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda^*)^+$ (the latter group consists of symmetric homomorphisms). Thus, for a field $F \supset \mathbb{Q}$ we can identify $\mathrm{NS}(A)\otimes F$ with the space of symmetric bilinear forms on $\Lambda\otimes F$. The ample cone in $\mathrm{NS}(A)\otimes\mathbb{R}$ consists of positive-definite forms. Thus, $D_A \subset \mathrm{NS}(A) \otimes \mathbb{R}$ is the Siegel's half-space consisting of symmetric bilinear forms on $\Lambda \otimes \mathbb{C}$ with positive-definite imaginary part.

According to [13] (see also [27, Sec. 6.5]), one can view the abelian variety B associated with an element $\omega = \omega_A \in D_A \simeq \mathfrak{H}_n$ as a mirror dual to (A, ω_A). More precisely, let us set

$$\Gamma_B = \Lambda^* \oplus \Lambda, \quad B = \Gamma_B \otimes \mathbb{R}/\Gamma_B,$$

and define the complex structure on $\Gamma_B \otimes \mathbb{R}$ via the isomorphism

$$\kappa_\omega : \Gamma_B \otimes \mathbb{R} \to \Lambda^* \otimes \mathbb{C} : (\lambda^*, \lambda) \mapsto \lambda^* - \omega(\lambda), \tag{3.5.1}$$

where we view ω as an element of $\mathrm{Hom}(\Lambda, \Lambda^* \otimes \mathbb{C})^+$. Note that there is an isomorphism $B \simeq \Lambda^* \otimes \mathbb{C}/(\Lambda^* + \omega\Lambda)$ (however, the corresponding identification of $H_1(B, \mathbb{Z})$ with Γ_B differs from the original one by the sign on the summand $\Lambda \subset \Gamma_B$). We have a natural principal polarization $\phi_0 : B\widetilde{\to}\hat{B}$ given on homology lattices by

$$\Gamma_B \to \Gamma_B^* : (\lambda_0^*, \lambda_0) \mapsto ((\lambda^*, \lambda) \mapsto \lambda^*(\lambda_0) - \lambda_0^*(\lambda). \tag{3.5.2}$$

Similarly, the natural isomorphism $\Lambda^* \oplus \Lambda^* \simeq \Gamma_{\hat{A}} \simeq \Gamma_A^*$ corresponds to the pairing

$$(\Lambda^* \oplus \Lambda^*) \times \Gamma_A \to \mathbb{Z} : ((\lambda_1^*, \lambda_2^*), (\lambda_1, \lambda_2)) \mapsto \lambda_1^*(\lambda_2) - \lambda_2^*(\lambda_1).$$

Let us define an isomorphism of orthogonal lattices

$$\gamma : \Gamma_A \oplus \Gamma_{\hat{A}} \to \Gamma_B \oplus \Gamma_B \simeq \Gamma_B \oplus \Gamma_{\hat{B}} : (\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*) \mapsto (\lambda_2^*, \lambda_2, \lambda_1^*, \lambda_1).$$

Proposition 3.5.1. *The isomorphism γ induces a mirror duality in the sense of* [13, Sec. 9] *between the pairs (A, ω_A) and (B, ω_B) for $\omega_B = \tau \cdot \phi_0$, where $\phi_0 \in \mathrm{Hom}(B, \hat{B})^+$ is the principal polarisation defined above.*

Proof. By definition, we have to check that the operator of complex structure on $(\Gamma_B \oplus \Gamma_B) \otimes \mathbb{R}$ corresponds under γ to

$$I_{\omega_A} = \begin{pmatrix} \alpha^{-1}\beta & -\alpha^{-1} \\ \alpha + \beta\alpha^{-1}\beta & -\beta\alpha^{-1} \end{pmatrix} \in \mathbf{U}_A(\mathbb{R}),$$

where $\omega_A = i\alpha+\beta$, and we view $\mathbf{U}_A(\mathbb{R})$ as a subgroup in automorphisms of $(\Gamma_A\oplus\Gamma_{\hat{A}})\otimes\mathbb{R}$, and similarly, that the complex structure on $(\Gamma_A \oplus \Gamma_{\hat{A}}) \otimes \mathbb{R}$ corresponds to I_{ω_B}. Both facts are checked by a straightforward computation (cf. [13, Prop. 9.6.1]). ☐

Recall that the variety $\mathbf{LG}_A = \mathbf{LG}_{E^n}$ is naturally identified with the Lagrangian Grassmannian associated with the symplectic lattice $\Lambda^* \oplus \Lambda$. Thus, a Lagrangian subvariety $L \subset A \times \hat{A}$, viewed as a point of $\mathbf{LG}(\mathbb{Q})$, corresponds to a Lagrangian $\mathbb{Z}$-submodule $\Pi(L) \subset \Lambda^* \oplus \Lambda = \Gamma_B$, so that $\Gamma_L = H_1(L, \mathbb{Z}) \simeq \Pi(L) \oplus \Pi(L) \subset \Gamma_A \oplus \Gamma_{\hat{A}}$. Hence, from (3.2.3) we get

$$\delta(L)(\omega) = \det(\kappa_\omega|_{\Pi(L)})^2 \bmod \mathbb{R}_{>0}, \tag{3.5.3}$$

where we view $\kappa_\omega|_{\Pi(L)}$ as an element in $\mathrm{Hom}(\Pi(L), \Lambda^*) \otimes \mathbb{C}$ and define $\det^2$ using some bases in $\Pi(L)$ and Λ^*.

Similarly, a point L of $\mathbf{LG}_A(\mathbb{R})$ corresponds to a real Lagrangian subspace $\Pi_{\mathbb{R}}(L) \subset \Gamma_B \otimes \mathbb{R}$ and the formula (3.5.3) still holds (with $\Pi_{\mathbb{R}}(L)$ instead of $\Pi(L)$). Recall that we have a double covering $\mathbf{LG}^{\mathrm{spin}}(A, \mathbb{R}) \to \mathbf{LG}(\mathbb{R})$ consisting of pairs $(L, \varphi) \in \mathbf{LG}_A(\mathbb{R}) \times \mathcal{O}(D_A)/\mathbb{R}_{>0}$ such that $\varphi^2 = \delta(L)$, so that the group $\mathrm{Spin}(\mathbb{R})$ acts on $\mathbf{LG}^{\mathrm{spin}}(A, \mathbb{R})$ (see Def. 3.2.4). In our case there is a splitting $\mathbf{U}(\mathbb{R}) \to \mathrm{Spin}(\mathbb{R})$ (see Remark 2.3.8.1), so we have an action of $\mathbf{U}(\mathbb{R})$ on $\mathbf{LG}^{\mathrm{spin}}(A, \mathbb{R})$ given by

$$g \cdot (L, \varphi) = (gL, \varphi'), \text{ where } \varphi'(g(\omega)) = \varphi(\omega) \cdot \det(a + b\omega)^{-1}.$$

We claim that $\mathbf{LG}^{\mathrm{spin}}(A, \mathbb{R}) \to \mathbf{LG}_A(\mathbb{R})$ is in fact the natural double covering corresponding to a choice of orientation on a Lagrangian subspace in $\Gamma_B \otimes \mathbb{R}$. Indeed, let us fix an orientation $\epsilon \in \bigwedge^n(\Lambda)$. Then a choice of a square root $\varphi = \sqrt{\delta(L)} \in \mathcal{O}^*(D_A)/\mathbb{R}_{>0}$ for $L \in \mathbf{LG}_A(\mathbb{R})$ induces an orientation on $\Pi_{\mathbb{R}}(L) \subset \Gamma_B \otimes \mathbb{R}$ as follows. By formula (3.5.3), for each ω the non-zero element

$$\varphi(\omega)^{-1} \cdot \det(\kappa_\omega|_{\Pi_{\mathbb{R}}(L)}) \in \bigwedge^n(\Lambda^*) \otimes \bigwedge^n(\Pi_{\mathbb{R}}(L))^{-1} \otimes_{\mathbb{R}} \mathbb{C},$$

depending continuously on ω, belongs to the $\mathbb{R}$-subspace $\bigwedge^n(\Lambda^*) \otimes \bigwedge^n(\Pi_{\mathbb{R}}(L))^{-1}$. Thus, we get an isomorphism

$$\bigwedge^n(\Pi_{\mathbb{R}}(L)) \simeq \bigwedge^n(\Lambda)^{-1} \otimes \mathbb{R}$$

and we define the orientation $\mu_{\varphi,\epsilon} \in \bigwedge^n(\Pi_{\mathbb{R}}(L))$ so that it corresponds to ϵ^{-1} under this isomorphism, i.e.,

$$\varphi(\omega)^{-1} \cdot \det(\kappa_\omega|_{\Pi_{\mathbb{R}}(L)}) \cdot \mu_{\varphi,\epsilon} = \epsilon^{-1}.$$

Let us associate with $L \in \mathbf{LG}_A(\mathbb{Q})$ the real subtorus in B by setting

$$T_L = \Pi(L) \otimes \mathbb{R}/\Pi(L) \subset \Gamma_B \otimes \mathbb{R}/\Gamma_B = B.$$

Note that T_L is Lagrangian with respect to the translation-invariant symplectic structure on B corresponding to the standard symplectic structure on $\Gamma_B = \Lambda^* \oplus \Lambda$ (i.e., this symplectic structure on B comes from the principal polarization ϕ_0). As we have shown above, a lifting of L to a point $(L, \varphi) \in \mathbf{LG}^{\mathrm{spin}}(A, Q)$ gives rise to an orientation of T_L.

Since $\mathbf{LG}^{\mathrm{spin}}(A, \mathbb{Q}) = \widetilde{\mathbf{LG}_A(\mathbb{Q})}/2\mathbb{Z}$, the map (3.2.4) induces a map

$$\overline{\mathrm{SH}}^{LI} \to \mathbf{LG}^{\mathrm{spin}}(A, \mathbb{Q}).$$

By Lemma 3.2.5, the composition

$$\mathrm{NS}(A) \otimes \mathbb{Q} \to \overline{\mathrm{SH}}^{LI} \to \mathbf{LG}^{\mathrm{spin}}(A, \mathbb{Q}) : \phi \mapsto V_\phi \mapsto (\Gamma(\phi), \varphi)$$

corresponds to the choice of the square root $\varphi(\omega) = \det(\phi - \omega)$, where we use dual bases of Λ and Λ^* to compute the determinant (see also Ex. 3.2.6). The corresponding orientation on $T_{\Gamma(\phi)}$ is induced by the isomorphism $\Pi(\Gamma(\phi)) \otimes \mathbb{R} \simeq \Lambda \otimes \mathbb{R}$ and the orientation ϵ of $\Lambda \otimes \mathbb{R}$.

Let $\Omega_{\omega,\epsilon}$ denote the holomorphic volume form on B defined by

$$\Omega_{\omega,\epsilon} = \kappa_\omega^*(\epsilon),$$

where we view $\epsilon \in \bigwedge^n(\Lambda) \subset \bigwedge^n(\Lambda) \otimes \mathbb{C}$ as an n-form on $\Lambda^* \otimes \mathbb{C}$ and use an isomorphism (3.5.1).

Theorem 3.5.2. *For an endosimple LI-object $F \in D^b(A)$ one has*

$$\chi(\ell(\omega), [F]) = \int_{[T_L]} \Omega_{\omega,\epsilon} \tag{3.5.4}$$

where $L = L_F$, and T_L is equipped with the orientation $\mu_{\varphi_F,\epsilon}$ coming from the point $(L_F, \varphi_F) \in \mathbf{LG}^{\mathrm{spin}}(A, \mathbb{Q})$ associated with F.

Proof. Note that shifting F by [1] changes the orientation of T_L, so the assertions for F and $F[n]$ are equivalent.

First, let us prove (3.5.4) in the case when L_F is transversal to $\{0\}\times\hat{A}$, i.e., when $F = V_\phi$ is the semihomogeneous bundle corresponding to $\phi \in \mathrm{Hom}(A, \hat{A})^+ \otimes \mathbb{Q} \simeq \mathrm{Hom}(\Lambda, \Lambda^*)^+ \otimes \mathbb{Q}$ (and $L_F = \Gamma(\phi) \subset A \times \hat{A}$). Recall that $\mathrm{rk}\, V_\phi = \deg(L_F \to A)^{1/2}$ (see (2.1.9)). For $K = \mathbb{Q}$ or $\mathbb{R}$ let $\Gamma_K(\phi) \subset (\Lambda^* \oplus \Lambda) \otimes K$ be the graph of ϕ viewed as a map of K-vector spaces (i.e., $\Gamma_K(\phi) = H_1(L_F, K)$) and set

$$\Gamma_{\mathbb{Z}}(\phi) := \Gamma_{\mathbb{Q}}(\phi) \cap (\Lambda^* \oplus \Lambda),$$

so that $T_L = \Gamma_{\mathbb{R}}(\phi)/\Gamma_{\mathbb{Z}}(\phi)$. We also denote by $i_\phi : \Gamma_{\mathbb{Z}}(\phi) \to \Lambda^* \oplus \Lambda$ the natural embedding. The orientation on T_L is induced by the natural isomorphism $\Gamma_{\mathbb{R}}(\phi) \simeq \Lambda \otimes \mathbb{R}$ and by the orientation of $\Lambda \otimes \mathbb{R}$ given by ϵ. The cycle $[T_L]$ in $H_n(A) \simeq \Lambda^n(\Lambda^* \oplus \Lambda)$ is the image of the positive generator $\mu \in \bigwedge^n(\Gamma_{\mathbb{Z}}(\phi))$ under the map

$$\bigwedge^n(i_\phi) : \bigwedge^n(\Gamma_{\mathbb{Z}}(\phi)) \to \bigwedge^n(\Lambda^* \oplus \Lambda).$$

Note also that the integration map

$$H_n(A) \to \mathbb{C} : \gamma \mapsto \int_\gamma \Omega_{\omega,\epsilon}$$

is identified with

$$\bigwedge^n(\kappa_\omega) : \bigwedge^n(\Lambda^* \oplus \Lambda) \to \bigwedge^n(\Lambda^*) \otimes \mathbb{C} \simeq \mathbb{C},$$

where the last isomorphism is given by ϵ. Hence, $\int_{[T_L]} \Omega_{\omega,\epsilon} = \delta(\mu) \cdot \epsilon$, where $\delta \in \bigwedge^n(\Lambda^*) \otimes \bigwedge^n(\Gamma_{\mathbb{Z}}(\phi))^{-1} \otimes \mathbb{C}$ is the determinant of the composition

$$\Gamma_{\mathbb{Z}}(\phi) \xrightarrow{i_\phi} \Lambda^* \oplus \Lambda \xrightarrow{\kappa_\omega} \Lambda^*.$$

The projection $p_2 : \Gamma_{\mathbb{Z}}(\phi) \to \Lambda$ is an embedding of index $\deg(L_F \to A)^{1/2}$, so the commutative diagram

$$\begin{tikzcd}
\Gamma_{\mathbb{Z}}(\phi) \arrow[r, "\kappa_\omega i_\phi"] \arrow[d, "p_2"'] & \Lambda^* \otimes \mathbb{C} \arrow[d, "\mathrm{id}"'] \\
\Lambda \arrow[r, "\phi - \omega"] & \Lambda^* \otimes \mathbb{C}
\end{tikzcd}$$

implies that

$$\delta(\mu)\cdot\epsilon = \det(\phi-\omega)\cdot\deg(L_F\to A)^{1/2} = \chi(\ell(\omega),\ell(\phi))\cdot \mathrm{rk}(F) = \chi(\ell(\omega),[F]),$$

where the last equality follows from Lemma 2.5.2.

Next, we will check that (3.5.4) is compatible with the action of the group $\mathbf{U}(\mathbb{Z})$ on $[F]$, ω and on B, where we use the natural symplectic action of $\mathbf{U}(\mathbb{Z})$ on $\Gamma_B = \Lambda\oplus\Lambda^*$ and the splitting $\mathbf{U}(\mathbb{Z})\to\mathbf{U}(\mathbb{Z})^{\mathrm{spin}}$ of the spin-covering (see Remark 2.3.8.1). Namely, for $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathbf{U}(\mathbb{Z})$ the relation

$$\begin{pmatrix}-\omega & \mathrm{id}_{\Lambda^*}\end{pmatrix}\cdot g^{-1} = (a+b\omega)^*\cdot\begin{pmatrix}-g(\omega) & \mathrm{id}_{\Lambda^*}\end{pmatrix}$$

leads to a commutative diagram

$$\begin{tikzcd} \Gamma_B \arrow[r, "\kappa_\omega"] \arrow[d, "g"'] & \Lambda^*\otimes\mathbb{C} \\ \Gamma_B \arrow[r, "\kappa_{g(\omega)}"] & \Lambda^*\otimes\mathbb{C} \arrow[u, "(a+b\omega)^*"'] \end{tikzcd} \tag{3.5.5}$$

Hence, we have

$$g^*\Omega_{g(\omega),\epsilon} = \det(a+b\omega)^{-1}\cdot\Omega_{\omega,\epsilon},$$

which implies that

$$\int_{g[T_L]}\Omega_{g(\omega),\epsilon} = \det(a+b\omega)^{-1}\cdot\int_{[T_L]}\Omega_{\omega,\epsilon}.$$

Also, the diagram (3.5.5) gives the equation

$$\kappa_\omega|_{\Pi(L)} = (a+b\omega)^*\circ\kappa_{g(\omega)}|_{\Pi(gL)}\circ g|_{\Pi(L)} \tag{3.5.6}$$

in $\mathrm{Hom}(\Pi(L),\Lambda^*)\otimes\mathbb{C}$. Since $g\cdot(L,\varphi) = (gL,\varphi')$, where $\varphi'(g(\omega)) = \varphi(\omega)\det(a+b\omega)^{-1}$, passing to determinants in (3.5.6) we obtain that the orientation $\mu_{\varphi',\epsilon}$ of $\Pi(gL)\otimes\mathbb{R}$ corresponds to $\mu_{\varphi,\epsilon}$ under the isomorphism $\Pi(L)\to\Pi(gL)$ given by g. Hence, the class $g[T_L]$ is exactly the fundamental class of T_{gL} associated with the orientation coming from $g[F]$. On the other hand, by Corollary 2.5.6,

$$\chi(\ell(\omega),[F]) = \chi(\ell(g(\omega)),g[F]),$$

since for $g\in\mathbf{U}(\mathbb{Z})$ the operator $\hat{\rho}(g)$ is simply the map induced by any autoequivalence of $D^b(A)$ compatible with the canonical lifting of g to $\mathbf{U}(\mathbb{Z})^{\mathrm{spin}}$.

Finally, applying Proposition 1.4.1 and using the $\mathbf{U}(\mathbb{Z})$-invariance, we see that the general case of (3.5.4) follows from the case when L_F is transversal to $\{0\}\times\hat{A}$ considered above. □

Remark 3.5.3. Note that since L_F is equipped with the lifting $\widetilde{L}_F$ to the universal covering of the Lagrangian Grassmannian (see Ex. 3.2.6), the Lagrangian torus T_L has a structure of a *graded Lagrangian* (see [33]). The corresponding choice of a phase of $\int_{T_L}$ obtained from Theorems 3.3.2 and 3.5.2 comes from Kontsevich's description of a grading on a Lagrangian (see [33, Ex. 2.9]).

4. Quasi-standard t-structures and Fourier-Mukai partners

4.1. **Quasi-standard t-structures.** The $\mathbb{Z}$-covering $\widetilde{\mathbf{LG}(\mathbb{Q})} \to \mathbf{LG}(\mathbb{Q})$ appears also naturally when considering t-structures. Let $\mathcal{T}(A)$ be the set of $\mathbf{H}$-invariant t-structures on $D^b(A)$. We identify $\mathcal{T}(A)$ with the set of cores of such t-structures, so we view elements of $\mathcal{T}(A)$ as abelian subcategories $\mathcal{A} \subset D^b(A)$.

Theorem 4.1.1. *(i) There is a natural $\widetilde{\mathbf{U}(\mathbb{Q})}$-equivariant embedding*

$$\widetilde{\mathbf{LG}(\mathbb{Q})} \to \mathcal{T}(A) : \widetilde{L} \mapsto \mathcal{A}_{\widetilde{L}},$$

which is uniquely characterized by the condition

$$\mathcal{A}_{(0:\phi_0),0} = \mathrm{Coh}(A).$$

The LI-functor $\Phi_{\widetilde{g}} : D^b(A) \to D^b(A)$ corresponding to $\widetilde{g} \in \widetilde{\mathbf{U}(\mathbb{Q})}$ (defined up to $\mathbf{H}$—see Sec. 2.1) satisfies

$$\Phi_{\widetilde{g}}(\mathcal{A}_{\widetilde{L}}) \subset \mathcal{A}_{\widetilde{g}\widetilde{L}}.$$

(ii) For an LI-object F and $\widetilde{L} \in \widetilde{\mathbf{LG}(\mathbb{Q})}$ one has $F[-i(\widetilde{L}_F, \widetilde{L})] \in \mathcal{A}_{\widetilde{L}}$, where $i(\cdot,\cdot) \in \mathbb{Z}$ is the unique $\widetilde{\mathbf{U}(\mathbb{Q})}$-equivariant function on $\widetilde{\mathbf{LG}(\mathbb{Q})} \times \widetilde{\mathbf{LG}(\mathbb{Q})}$ such that for $\phi_1, \phi_2 \in \mathrm{NS}(A) \otimes \mathbb{Q}$ one has

$$i(\widetilde{L}_{V_{\phi_1}}, \widetilde{L}_{V_{\phi_2}}) = i(\phi_2 - \phi_1),$$

provided $\phi_2 - \phi_1$ is nondegenerate (recall that $\widetilde{L}_{V_\phi}$ is given by (3.2.5)*).*

Proof. (i) Recall that the action of $\widetilde{\mathbf{U}(\mathbb{Q})}$ on $\widetilde{\mathbf{LG}(\mathbb{Q})}$ is transitive, and the stabilizer subgroup of the point $((0 : \phi_0), 0)$ is $\mathbf{P}^-(\mathbb{Q})$, lifted to $\widetilde{\mathbf{U}(\mathbb{Q})}$ as described in Corollary 2.2.2. Thus, it suffices to check that $\mathbf{P}^-(\mathbb{Q})$ preserves the standard t-structure. But this immediately follows from the description of the functors corresponding to elements of $\mathbf{P}^-(\mathbb{Q})$ (see Prop. 2.2.1).
(ii) The fact that every LI-sheaf is cohomologically pure with respect to each t-structure constructed in (i) follows from Theorem 2.4.1. Uniqueness of the $\widetilde{\mathbf{U}(\mathbb{Q})}$-equivariant index function $i(\cdot,\cdot)$ follows from Proposition 1.4.1. It remains to find the number $i = i(\phi_1, \phi_2)$ such that

$$V_{\phi_1}[-i] \in \mathcal{A}_{\widetilde{L}_{V_{\phi_2}}}.$$

Let $g = \begin{pmatrix} 1 & \phi_2^{-1} \\ 0 & 1 \end{pmatrix}$. Then by formula (2.4.1), we have

$$\Phi_g(\mathcal{O}_x) = V_{\phi_2} \bmod \mathbb{N}^*$$

(there is no shift in this case since the kernel $S(g)$ is a vector bundle). It follows that

$$\Phi_g(\mathrm{Coh}(A)) \subset \mathcal{A}_{\widetilde{L}_{V_{\phi_2}}}.$$

Note that $\Gamma(\phi_1) = g(\Gamma(\phi))$, where

$$\phi = \phi_1(1 - \phi_2^{-1}\phi_1)^{-1}.$$

Hence, using (2.4.1) and (2.4.3) we obtain

$$\Phi_g(V_\phi) = V_{\phi_1}[-i(\phi_2 + \phi)] \bmod \mathbb{N}^*,$$

so denoting $\phi_3 = 1 - \phi_2^{-1}\phi_1$ we obtain

$$i = i(\phi_2 + \phi_1\phi_3^{-1}) = i(\phi_3(\phi_2\phi_3 + \phi_1)) = i(\phi_3\phi_2) = i(\phi_2 - \phi_1)$$

as claimed. □

Definition 4.1.2. We will refer to t-structures on $D^b(A)$ constructed in the above theorem as *quasi-standard t-structures.*

Proposition 4.1.3. *Let A and B be abelian varieties, and let $\eta : X_A \to X_B$ be a symplectic isomorphism in $\mathcal{A}b_{\mathbb{Q}}$ (i.e., up to isogeny). Then the map $\eta_* : \mathbf{LG}_A(\mathbb{Q}) \to \mathbf{LG}_B(\mathbb{Q})$ induced by η extends to a $\mathbb{Z}$-equivariant map*

$$\widetilde{\eta}_* : \widetilde{\mathbf{LG}_A(\mathbb{Q})} \to \widetilde{\mathbf{LG}_B(\mathbb{Q})}$$

which is compatible with the quasi-standard t-structures, i.e., for every $\widetilde{L} \in \widetilde{\mathbf{LG}_A(\mathbb{Q})}$ the LI-functor Φ_η associated with η (defined up to $\mathbf{H}$) satisfies

$$\Phi_\eta(\mathcal{A}_{\widetilde{L}}) \subset \mathcal{A}_{\widetilde{\eta}_*\widetilde{L}}. \tag{4.1.1}$$

Proof. Note that B is isogenous to A, i.e., there exists an isomorphism $f : A \to B$ in $\mathcal{A}b_{\mathbb{Q}}$. Let $\eta_0 : X_A \to X_B$ be the induced symplectic isomorphism in $\mathcal{A}b_{\mathbb{Q}}$. We also have natural compatible isomorphisms induced by f:

$$\mathbf{U}_{X_A} \to \mathbf{U}_{X_B}, \quad \widetilde{\mathbf{U}_{X_A}(\mathbb{Q})} \to \widetilde{\mathbf{U}_{X_B}(\mathbb{Q})}$$

$$\eta_{0*} : \mathbf{LG}_A(\mathbb{Q}) \to \mathbf{LG}_B(\mathbb{Q}), \quad \widetilde{\eta}_{0*} : \widetilde{\mathbf{LG}_A(\mathbb{Q})} \to \widetilde{\mathbf{LG}_B(\mathbb{Q})}.$$

Furthermore, it is easy to see that the t-exactness (4.1.1) holds for $\widetilde{\eta}_{0*}$ and the functor Φ_{η_0} which is the composition of the pull-back and the push-forward under isogenies (this is proved similarly to Prop. 2.2.1(ii)). Now let $g_\eta \in \mathbf{U}(\mathbb{Q})$ be the unique element such that

$$\eta = \eta_0 \circ g_\eta.$$

Choose any element $\widetilde{g}_\eta \in \widetilde{\mathbf{U}(\mathbb{Q})}$ over g_η and define

$$\widetilde{\eta}_* : \widetilde{\mathbf{LG}_A(\mathbb{Q})} \to \widetilde{\mathbf{LG}_B(\mathbb{Q})} : \widetilde{L} \mapsto \widetilde{\eta}_{0*}(\widetilde{g}_\eta(\widetilde{L})).$$

By Theorem 4.1.1(i), the required assertion follows for the functor $\Phi_{\eta_0} \circ \Phi_{\widetilde{g}_\eta}$. By [31, Thm. 3.2.11], its $\mathbf{H}$-equivalence class differs from $\Phi_\eta[n]$ by an action of $\mathbb{N}^*$ (one has to use also [31, Prop. 2.4.7(ii)] as in the proof of [31, Thm. 3.3.4]). Changing $\widetilde{\eta}_*$ using the action of $n \in \mathbb{Z} \subset \widetilde{\mathbf{U}(\mathbb{Q})}$ on $\widetilde{\mathbf{LG}_A(\mathbb{Q})}$, we get the required compatibility (4.1.1). □

Remarks 4.1.4. 1. The quasi-standard t-structure associated with $\widetilde{L}_F \in \widetilde{\mathbf{LG}(\mathbb{Q})}$ has a simple characterization in terms of the LI-object F (defined up to $\mathbf{H}$-equivalence). Namely, the corresponding subcategory $D^{\le 0} \subset D^b(A)$ consists of all $X \in D^b(A)$ such that $\mathrm{Hom}^i(X, T_{x,\xi}(F)) = 0$ for $i < 0$ and all $(x, \xi) \in A \times \hat{A}$. Indeed, using $\widetilde{\mathbf{U}(\mathbb{Q})}$-action this

reduces to the characterization of the standard subcategory $D^{\leq 0}$ by the above condition, where F is a nonzero torsion sheaf.
2. In the case of an elliptic curve all the quasi-standard t-structures are obtained from the standard one by tilting (up to a shift). More precisely, let $P(\cdot)$ be the slicing associated with the standard stability on $D^b(E)$ for an elliptic curve E, so that $P((0,1]) = \mathrm{Coh}(E)$ (see Ex. 3.3.5). Then the quasi-standard t-structure associated with $\phi \in \mathrm{NS}(E,\mathbb{Q}) \simeq \mathbb{Q}$ (lifted to $\widetilde{\mathbf{LG}}_E(\mathbb{Q})$ by (3.2.6)) is $P((\frac{\mathrm{Arg}(i-\phi)}{\pi} - 1, \frac{\mathrm{Arg}(i-\phi)}{\pi}])$. Note that this construction extends to irrational numbers $\phi \in \mathbb{R}$ and for $k = \mathbb{C}$ the corresponding hearts are equivalent to the categories of holomorphic bundles on noncommutative 2-tori (see [32], [28]). We conjecture that this connection between quasi-standard t-structures and noncommutative tori extends to the higher-dimensional case (the corresponding equivalence of derived categories is established in [4]). Namely, to every point of $\widetilde{\mathbf{LG}}_A(\mathbb{R}) \setminus \widetilde{\mathbf{LG}}_A(\mathbb{Q})$ there should correspond a t-structure on $D^b(A)$ (in a way compatible with the action of $\widetilde{\mathbf{U}(\mathbb{Q})}$) whose heart is equivalent to the category of holomorphic bundles on the corresponding noncommutative torus.

4.2. **Fourier-Mukai partners.** Recall that the set of Fourier-Mukai partners (*FM-partners* for short) of a smooth projective variety X is defined as

$$\mathrm{FM}(X) = \{Y \text{ smooth projective} \mid D^b(Y) \simeq D^b(X)\}/\text{isomorphism}.$$

For an abelian variety A we can also define the subset $\mathrm{FM}^{ab}(A) \subset \mathrm{FM}(A)$ by considering only FM-partners among abelian varieties. In characteristic zero it is known that $\mathrm{FM}^{ab}(A) = \mathrm{FM}(A)$ (see [16]).

Recall that if B is a FM-partner of A then any equivalence $D^b(A) \simeq D^b(B)$ is given by the LI-kernel associated with a Lagrangian correspondence $(L(\eta), \alpha)$ extending a symplectic isomorphism $\eta : X_A \simeq X_B$ (see 2.1). The $\mathbf{U}(\mathbb{Z})$-orbit of the Lagrangian $(\eta_*)^{-1}(0 \times \hat{B}) \in \mathbf{LG}_A(\mathbb{Q})$ does not depend on a choice of an equivalence $D^b(A) \simeq D^b(B)$.

Proposition 4.2.1. *The above construction gives an embedding*

$$\mathrm{FM}^{ab}(A) \hookrightarrow \mathbf{LG}_A(\mathbb{Q})/\mathbf{U}(\mathbb{Z}). \tag{4.2.1}$$

The image consists of orbits of Lagrangian subvarieties $L \subset X_A$ for which there exists a Lagrangian subvariety $L' \subset X_A$ such that $L \cap L' = 0$.

Proof. The first assertion is immediate since the Lagrangian subvariety $(\eta_*)^{-1}(0 \times \hat{B}) \subset X_A$ corresponding to B is isomorphic to $\hat{B}$. For the second we observe that if we have a Lagrangian $L' \subset X_A$ such that $L \cap L' = 0$ then the get an isomorphism $L \times L' \to X_A$ and also $L' \simeq X_A/L \simeq \hat{L}$, which leads to a symplectic isomorphism $L \times \hat{L} \simeq X_A$, so that $B = \hat{L}$ is a FM-partner of A. $\square$

Remark 4.2.2. The set $\mathbf{LG}_A(\mathbb{Q})/\mathbf{U}(\mathbb{Z}) = \mathbf{U}(\mathbb{Z})\backslash\mathbf{U}(\mathbb{Q})/\mathbf{P}^-(\mathbb{Q})$ is known to be finite (see [12, Thm. 6]). Note that this set is also in bijection with the set of endosimple LI-objects in $D^b(A)$ up to the action of exact autoequivalences of $D^b(A)$ (as follows from Prop. 2.1.2).

Here is an example of a situation when the embedding of Proposition 4.2.1 is a bijection.

Proposition 4.2.3. *Assume that A is principally polarized and* $\mathrm{End}(A) = R$ *is the ring of integers in a totally real number field F (so the Rosati involution on F is trivial). Then the map* (4.2.1) *is a bijection, and*

$$|\,\mathrm{FM}^{ab}(A)| = |\mathbf{LG}_A(\mathbb{Q})/\mathbf{U}(\mathbb{Z})| = h_R,$$

where h_R is the class number of R.

Proof. First, we observe that in this case the set $\mathbf{LG}_A(\mathbb{Q})$ consists of all subvarieties in $X_A = A \times A$, isogenous to A. We claim that all such subvarieties $L \subset X_A$ are direct summands. Indeed, L is an image of the morphism $A \to A^2$ associated with a pair $(a, b) \in R^2 \setminus \{(0, 0)\}$. Consider the exact sequence

$$0 \to I' \to R^2 \to I \to 0,$$

where $I = (a, b) \subset R$. This sequence splits since I is a projective R-module. Hence, there is a corresponding split exact sequence of abelian varieties

$$0 \to A^I \to A^2 \to A^{I'} \to 0,$$

where we use the natural functor $M \to A^M$ from R-modules to commutative group schemes with $A^M(S) = \mathrm{Hom}_R(M, A(S))$ (see [11]) Since A^I is exactly the image of the map $(a, b) : A \to A^2$, this proves our claim.

It remains to check that the orbits of $\mathrm{SL}_2(R)$ on the projective line $\mathbb{P}^1(F)$ are in bijection with the ideal class group $\mathrm{Cl}(R)$. We have a well defined map

$$\mathbb{P}^1(F)/\,\mathrm{SL}_2(R) \to \mathrm{Cl}(R)$$

sending $(a : b)$ with $a, b \in R$ to the class of the ideal (a, b). This map is surjective since every nonzero ideal in R is generated by two elements. To show injectivity suppose that pairs $(a : b)$ and $(a' : b')$ define the same ideal class. Then upon rescaling we can assume that $(a, b) = (a', b')$. Now we have two surjective maps $R^2 \to I = (a, b)$: one given by (a, b) and another by (a', b'), and our assertion follows from Lemma 4.2.4 below. □

Lemma 4.2.4. *For every nonzero ideal $I \subset R$ the action of* $\mathrm{SL}_2(R)$ *on surjective maps $R^2 \to I$ is transitive.*

Proof. Since I is a projective R-module, for every surjective map $f : R^2 \to I$ there exists an isomorphism

$$\alpha : R^2 \xrightarrow{\sim} I' \oplus I$$

such that f is the composition of α with the projection to I. Note that $\det(\alpha)$ induces an isomorphism of R with $I' \otimes_R I$, so we obtain an isomorphism $I' \simeq I^{-1}$. Thus, we can view α as an isomorphism $R^2 \to I^{-1} \oplus I$ such that $\det(\alpha)$ is the canonical isomorphism $R \to I^{-1} \otimes I$. If $g : R^2 \to I$ is another surjective morphism and $\beta : R^2 \xrightarrow{\sim} I^{-1} \oplus I$ is the corresponding isomorphism then $\gamma = \beta^{-1} \circ \alpha$ is an element of $\mathrm{SL}_2(R)$ such that $g \circ \gamma = f$. □

Remark 4.2.5. In general the embedding (4.2.1) is not a bijection as one can see already in the case of a non-principally polarized abelian variety with $\mathrm{End}(A) = \mathbb{Z}$ (cf. [23, Ex. 4.16]). The Lagrangians not in the image of this map correspond to categories of twisted

sheaves equivalent to $D^b(A)$ (see [25]). Note that the set $\mathbf{LG}_A(\mathbb{Q})$ is a subset of vertices of the spherical building associated with the group $\mathbf{U}$, which is related to the boundary of the Baily-Borel compactification of the Siegel domain D_A. It would be interesting to see whether other elements of this building have an interpretation in terms of $D^b(A)$. Also, one can expect some relation between the quasi-standard t-structures and the t-structures associated with stabilities coming from points of D_A or $D_A \times \mathbb{C}$. In the case of K3-surfaces similar questions are studied in [17] and [15].

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